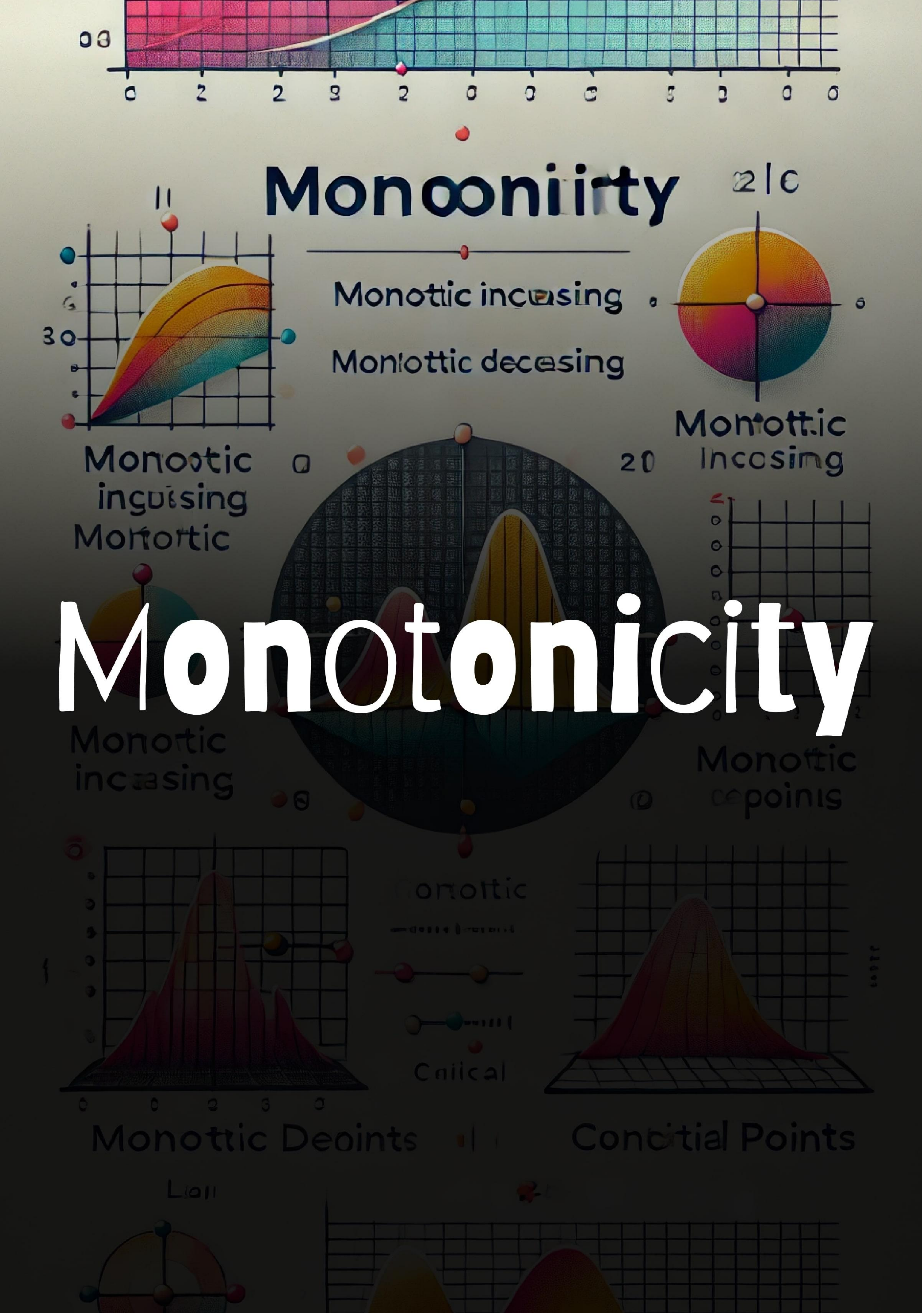


Monotonicity

MONOTONICITY

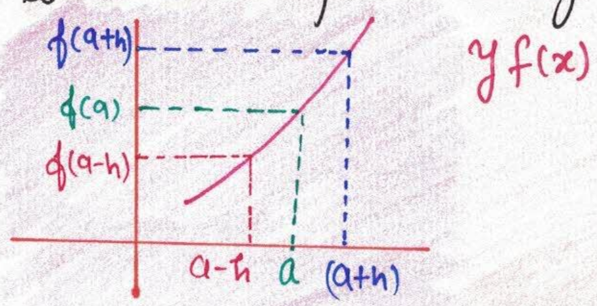
If $f(x)$ is → Strictly ↑ / Strictly ↓ / Non increasing / Non decreasing, and $f(x)$ satisfies any of these four properties then $f(x)$ is said to be monotonic, otherwise it is called nonmonotonic in it domain.

Monotonic at a Point $x = a$.

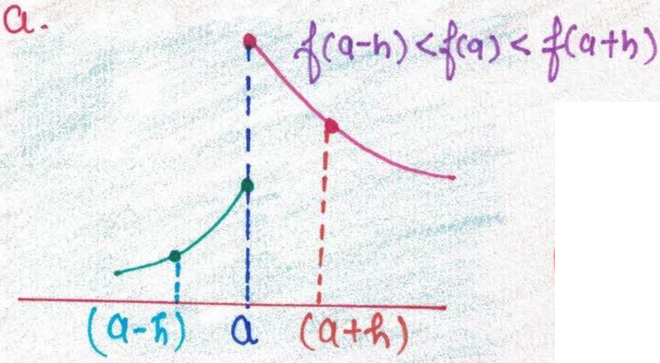
Let $f(x)$ is a function where domain is D and if $a \in D$, then:-

1. **STRICTLY INCREASING At a.** If $f(a-h) < f(a) < f(a+h)$ for sufficiently small +ve value of h, then $f(x)$ is said to be strictly increasing at a.

Ex



Strictly ↑ graph



Strictly ↑ graph

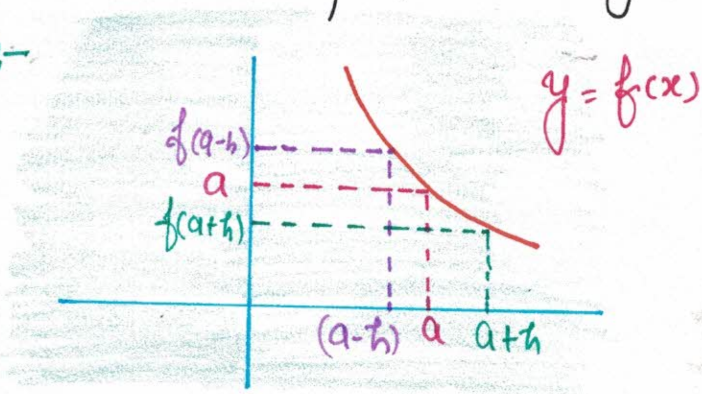
> "Continuity and Differentiability are not taken into consideration."

2. STRICTLY DECREASING AT a :-

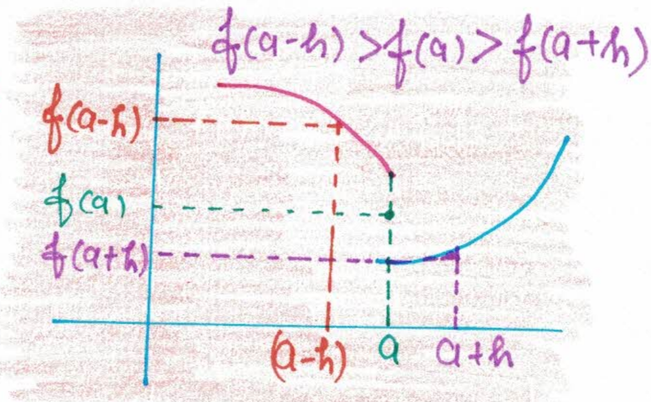
$$\boxed{\text{if } f(a-h) > f(a) > f(a+h)}$$

for sufficiently small +ve value of h, then $f(x)$ is strictly decreasing at $x=a$.

Ex:-



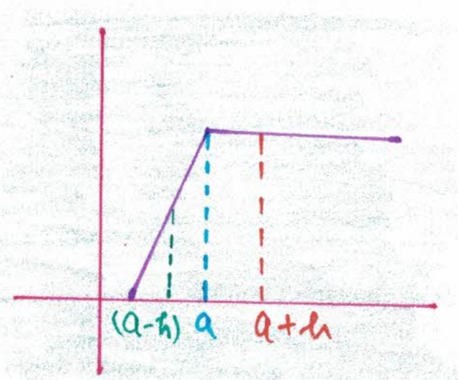
Strictly ↓



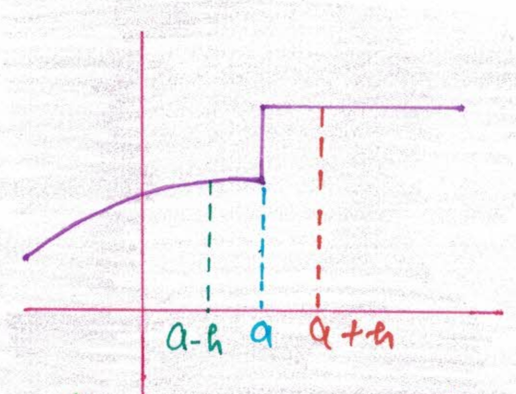
Strictly ↓ graph

3. NON DECREASING AT $x = a$

if $f(a-h) \leq f(a) \leq f(a+h)$, then $f(x)$ is said to be non decreasing at $x=a$ for small +ve value of h.



$f(a-h) < f(a) = f(a+h)$

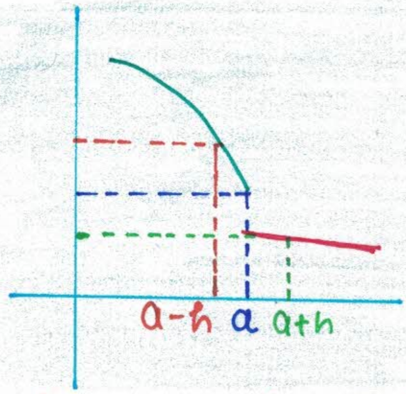


$f(a-h) < f(a) = f(a+h)$

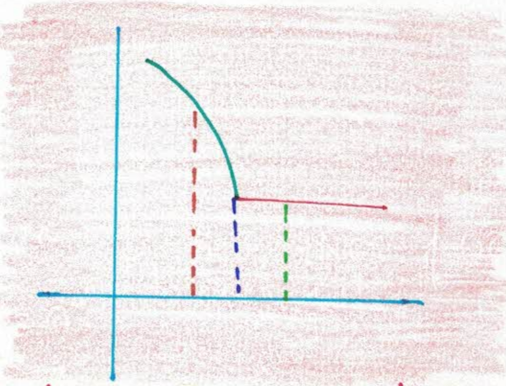
4. NON INCREASING AT $x=a$.

If $f(a-h) \geq f(a) \geq f(a+h)$, then $f(x)$ is said to be non increasing at $x=a$.

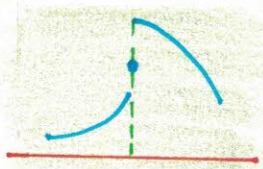
Ex:



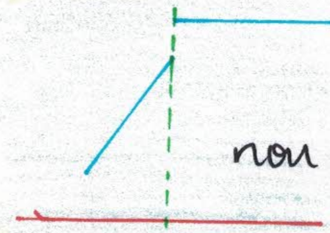
$f(a-h) \not> f(a) = f(a+h)$



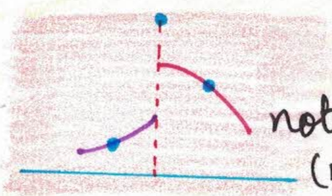
$f(a-h) > f(a) = f(a+h)$



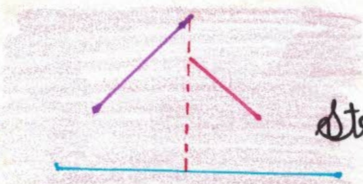
Strictly ↑



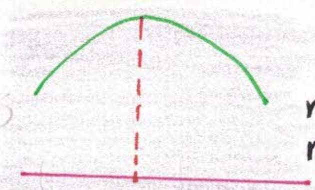
non decreasing



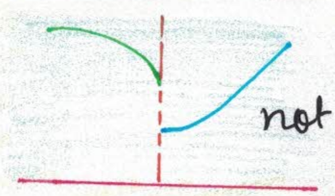
not defined (monotonous)



Strictly ↓



not monotonous



not monotonous

Necessary condition for f(x) to be monotonous at $x=a$.

If $f(x)$ is continuous and diff^n at $x = a \in D_f$, then

- If $f'(a) > 0$, then $f(x)$ is strictly ↑.
- If $f'(a) < 0$, then $f(x)$ is strictly ↓.
- If $f'(a) = 0$, then if –

1. If $f(a-h) > 0$ and $f'(a+h) > 0$, then $f(x)$ is strictly ↑ at $x=a$.
2. If $f'(a-h) < 0$ and $f'(a+h) < 0$, then $f(x)$ is strictly ↓ at $x=a$.
3. If $f'(a+h)$ and $f'(a-h)$ are of opposite sign then $f(x)$ is neither increasing non decreasing at $x=a$.

Ex:- check the behaviour of $f(x) = x^3 - 3x + 7$ at $x = 0, 1, 2$

$\rightarrow f'(x) = 3x^2 - 3$

$f'(0) = -3 < 0$ $\quad$ $f(0)$ is strictly $\downarrow$ at $x = 0$

$f'(1) = 0$

$f'(1+h) = 3(1+h)^2 - 3 \Rightarrow 3(1+h^2+2h) - 3 = 3h^2 + 6h > 0$

$f'(1-h) = 3(1-h)^2 - 3 \Rightarrow 3(1+h^2-2h) - 3 = 3h^2 - 6h < 0$

$f(x)$ is neither $\uparrow$ nor $\downarrow$ at $x = 1$

$f'(2) = 9 > 0$

$f(x)$ is strictly $\uparrow$ at $x = 2$

Ex:- $f(x) = e^{x^2}$ at $x = 0$

$f'(x) = 2xe^{x^2}$

$f'(0) = 0$

$f'(h) = 2he^{h^2} > 0 \Rightarrow f'(-h) = -2he^{+h^2} < 0$

Neither $\uparrow$ nor $\downarrow$ at $x = 0$.

Ex:- $f(x) = (x-2)^{5/3}$ at $x = 2$.

$f(x) = \frac{5}{3}(x-2)^{2/3} \Rightarrow \frac{5}{3}\left[\sqrt[3]{x-2}\right]^2$

$f'(2) = 0$

$f'(2) = 0$

$f'(2-h) = \frac{5}{3}\left(\sqrt[3]{2-h-2}\right)^2 = \frac{5}{3}(-h)^{2/3} = \frac{5}{3}(h)^{2/3} > 0$

$f'(2+h) = \frac{5}{3}\left(\sqrt[3]{2+h-2}\right)^2 = \frac{5}{3}(h)^{2/3} > 0$

Strictly $\uparrow$

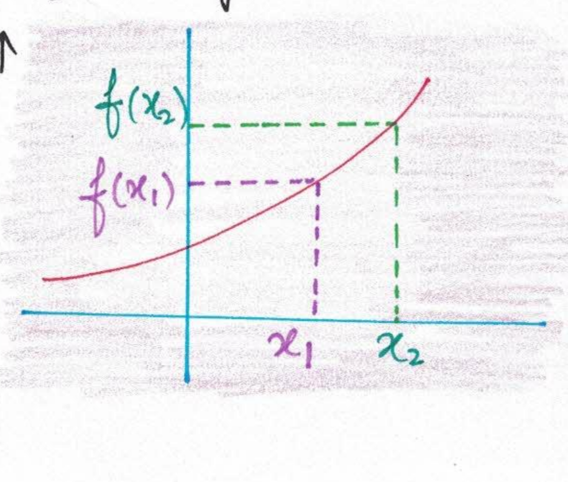
MONOTONICITY AT END POINT OF FUNCTION

Let $f(x)$ is defined for $x \in [a, b]$ and $f(x)$ is continuous and differentiable at $x = a$ & b, then –

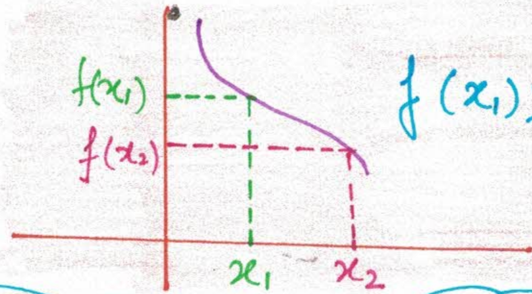
- If $f'(a^+) > 0$, then $f(x)$ is strictly ↑ at $x = a$.
- If $f'(a^+) < 0$, then $f(x)$ is strictly ↓ at $x = a$.
- if $f'(a^+) = 0$, then –
  - (a) If $f'(a+h) > 0$, then $f(x)$ is strictly ↑ at $x = a$.
  - (b) If $f'(a+h) < 0$, then $f(x)$ is strictly ↓ at $x = a$.
- If $f'(b^-) = 0$, then –
  - (a) If $f'(b-h) > 0$, then $f(x)$ is strictly ↑ at $x = b$.
  - (b) If $f'(b-h) < 0$, then $f(x)$ is strictly ↓ at $x = b$.
- If $f'(b^-) > 0$, then $f(x)$ is strictly ↑ at $x = b$
- If $f'(b^-) < 0$, then $f(x)$ is strictly ↓ at $x = b$.

MONOTONICITY IN AN INTERVAL

1. Let $f(x)$ is defined for $x \in D$ and consider $x_1, x_2 \in D$ such that $x_1 < x_2$ then if – $f(x_1) < f(x_2)$ holds true for every $x_1, x_2 \in D$ then $f(x)$ is said to strictly ↑

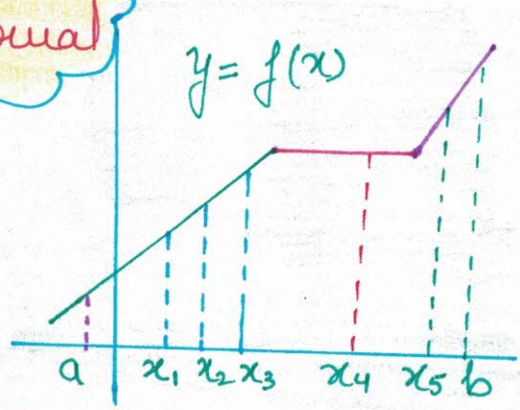


2 Strictly ↓ in an interval if $x_1 < x_2$ and $f(x_1) > f(x_2)$ $\forall\, x_1, x_2 \in D$ then $f(x)$ is said to be strictly ↓ in domain.



$f(x_1) > f(x_2)$

3 Non decreasing in an interval

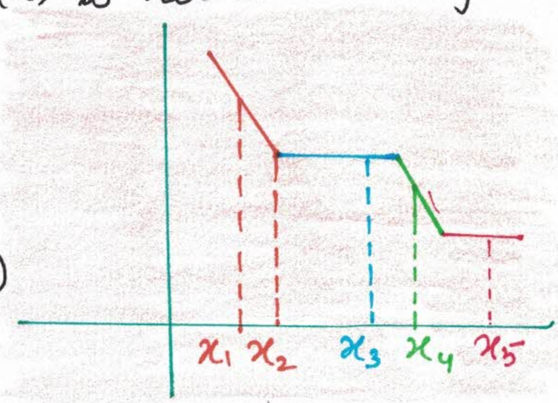


$x_1 < x_2 < x_3 < x_4 < x_5$

$f(x_1) < f(x_2) < f(x_3) = f(x_4) < f(x_5)$

4 NON INCREASING IN AN INTERVAL

Let $f(x)$ be continous and defined for $x \in [a, b]$ if $x_1, x_2 \in [a, b]$ such that $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ holds true $\forall\, x_1, x_2 \in [a, b]$ then $f(x)$ is non increasing in $[a, b]$.



$x_1 < x_2 < x_3 < x_4 < x_5$

$f(x_1) > f(x_2) = f(x_3) > f(x_4) = f(x_5)$

$f(x)$ is non increasing.

NECESSARY CONDITION FOR MONOTONICITY OUR AN INTERVAL

Let $f(x)$ be a continuous and diff^n for all $x \in [a, b]$ except some finite point in $[a, b]$.

1. If $f'(x) \geq 0 \ \forall x \in [a, b]$ then $f(x)$ is said to be strictly ↑ in $[a, b]$ and $f'(x) = 0$ for some sep-rated pts. | discrete pts. of $[a, b]$.

Ex:—

$$f(x) = x - \sin x$$
$$f'(x) = 1 - \cos x$$

$f'(x) \geq 0$ and $f'(x) = 0$ at some discrete points, hence it is strictly ↑ function.

2. If $f'(x) \leq 0 \ \forall x \in [a, b]$ and $f(x) = 0$ for some discrete pts. in $[a, b]$, then $f(x)$ is strictly ↓ for $[a, b]$.

3. If $f'(x) \geq 0$ and $f'(x) = 0$ for some sub interval $[c, d]$ of $[a, b]$ $f(x)$ is non decreasing in $[a, b]$.

4. If $f'(x) \leq 0$ and $f'(x) = 0$ for some sub interval $[c, d]$ of $[a, b]$, then $f(x)$ is non increasing in $[a, b]$.

Question

$$f(x) = \begin{cases} 1, & x \leq 0 \\ xe^{x+1}, & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} 0, & x \leq 0 \\ xe^{x} + e^{x}, & x > 0 \end{cases}$$

$f'(x) \geq 0$ but $f'(x) = 0$ for $x \in [-\infty, 0]$

it is non decreasing

Question

$f(x) = -x - \cot^{-1} x.$

$$f'(x) = -1 + \frac{1}{1+x^2} \Rightarrow \frac{-1-x^2+1}{1+x^2} = \frac{-x^2}{1+x^2}$$

$$\boxed{f'(x) \leq 0 \text{ , strictly } \downarrow}$$

Question

$f(x) = 2x - \cos x$

$$f'(x) = 2 + \sin x$$
$$= 2 + [-1, 1] \Rightarrow [1, 3] \geq 0 \quad \boxed{\text{Strictly } \uparrow}$$

Question

$$f(x) = \frac{x}{\sin x} \quad ; \; x \in (0, 1)$$

$$g(x) = \frac{x}{\tan x}$$

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \frac{1}{\sin x} - \frac{x\cos x}{\sin^2 x}$$

Let $\sin x - x \cos x = f_1(x)$

$$f_1'(x) = \cos x - \cos x + x \sin x$$
$$= x \sin x > 0$$

$f'(x) > 0 \Rightarrow f(x)$ is increasing

$$g(x) = \frac{x}{\tan x} \Rightarrow g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$$

Let $f_1(x) = \tan x - x \sec^2 x$

$$f_1'(x) = \sec^2 x - \sec^2 x - x \times 2\sec^2 x \tan x$$
$$= -2x \sec^2 x \tan x$$

$f_1'(x) < 0 \qquad g(x)$ is decreasing

Question

$$f(x) = \frac{x^2}{2-2\cos x} \qquad 0 < x < 1$$

$$g(x) = \frac{x^2}{6x - 6\sin x}$$

$$f'(x) = \frac{(2-2\cos x) \times 2x - x^2 (0 + 2\sin x)}{(2-2\cos x)^2}$$

$$= \frac{4x - 4x\cos x - 2x^2 \sin x}{(2-2\cos x)^2}$$

Let $f_1(x) = 4x - 4x\cos x - 2x^2 \sin x.$

$$f_1'(x) = 4 + 4x\sin x - 4\cos x - 2x^2\cos x - 2\times 2x\sin x$$

$$= 4 + 4x\sin x - 4\cos x - 2x^2\cos x - 4x\sin x$$

$$= 4 - 4\cos x - 2x^2\cos x$$

$$= 4 - 2\cos x(2 + x^2) > 0$$

It is strictly $\uparrow$.

$$g'(x) = \frac{(6x - 6\sin x)(2x) - x^2(6 - 6\cos x)}{(6x - 6\sin x)^2}$$

$$= \frac{12x^2 - 12x\sin x - 6x^2 + 6x^2\cos x}{(6x - 6\sin x)^2}$$

Let $f_1(x) = 6x^2 + 6x^2\cos x - 12x\sin x.$

$$= 6x^2 + (1 + \cos x) - 12x\sin x.$$

Let $g_1(x) = 6x^2(1 + \cos x) - 12x\sin x$

$$g_1'(x) = 6x^2(-\sin x) + 12x(1 + \cos x) - 12x\cos x - 12\sin x$$

$$= -6x^2\sin x + 12x + 12x\cos x - 12x\cos x + 2\sin x$$

$$= -6x^2\sin x - 12\sin x + 12x$$

$$= \underset{\downarrow}{-6x^2\sin x} - 12\underset{\downarrow}{(\sin x - x)}$$

$$\ominus \qquad -12\,[(0, \sin 57, 3) - (0,1)]$$

$= \ominus \quad -12(0, \sin 54, -1)$

$g(x)$ is decreasing

Check $f'(x)$, not $f(x)$

Question $f(x) = x \tan^{-1} x \ \forall \ x \in R.$

$\rightarrow f'(x) = \dfrac{x}{1+x^2} + \tan^{-1} x \Rightarrow$ $x<0, \ f'(x)<0$

$x>0, \ f'(x)>0$

$\boxed{\text{Non Monotonous}}$

Question $f(x) = \sin(\cos x) \ \forall \ x \in \left(\pi, \frac{3\pi}{2}\right)$

$\rightarrow f'(x) = -\left[\underbrace{\cos(\cos x)}_{\oplus} \underbrace{\sin x}_{\ominus}\right] = \oplus$

$f'(x) > 0$ $\boxed{\text{It is strictly } \uparrow}$

Question If $f(x) = \sin x - bx + c$ is strictly $\uparrow \ \forall \ x \in R$ find b.

$\rightarrow f(x) = \sin x - bx + c$

$f'(x) = \cos x - b$

$f'(x) \geq 0$

$\cos x - b \geq 0 \Rightarrow$ $\boxed{b \leqslant \cos x \Rightarrow b \in [-\infty, -1]}$

Question If $f(x) = x^2 + kx + 2014$ is strictly $\uparrow$ for $x \in [1,2]$ Then find lowest absolute value of K.

$\rightarrow f(x) = x^2 + kx + 2014$

$f'(x) = 2x + k$

$f'(x) \geq 0$

$2x+k \geq 0$

$k \geq -2x \qquad -2x \in [-4,-2]$

$k \geq [-4,-2] \Rightarrow k \geq -2 \qquad \boxed{|k| = 2}$

Question

If $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x + \ln 5$ is strictly ↓ $\forall x \in R$ then find a.

$\rightarrow f'(x) = 5\left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^4 - 3$

$5\left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^4 - 3 \leq 0$

$\left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^4 \leq 3/5 \Rightarrow \left(\frac{\sqrt{a+4}}{1-a} - 1\right) < \frac{3}{5x^4}$

$\left(\frac{\sqrt{a+4}}{1-a} - 1\right) \leq 0 \Rightarrow \frac{\sqrt{a+4}}{1-a} \leq 1$

CASE-I

$1-a > 0$

$a+4 \leq (1-a)^2 \Rightarrow a+4 \leq 1+a^2-2a$

$a^2-3a-3 \geq 0$

$a^2-3a-3 \geq 0 \Rightarrow a = \frac{3 \pm \sqrt{9+12}}{2}$

$a = \frac{3 \pm \sqrt{21}}{2}$

$a \in \left[\frac{3+\sqrt{21}}{2}, \infty\right] \cup \left(-\infty, \frac{3-\sqrt{21}}{2}\right]$

$1-a > 0 \Rightarrow a < 1 \qquad a \in (-\infty, 1)$

$a \in \left(-\infty, \frac{3-\sqrt{21}}{2}\right] \cap [-4, \infty) = \left[-4, \frac{3-\sqrt{21}}{2}\right]$

CASE II

$1 - a < 0$

$a > 1 \qquad a \in (1, \infty)$

$$\frac{\sqrt{a+4}}{1-a} \leq 1 \Rightarrow \sqrt{a+4} \geq 1-a$$

$$\downarrow$$

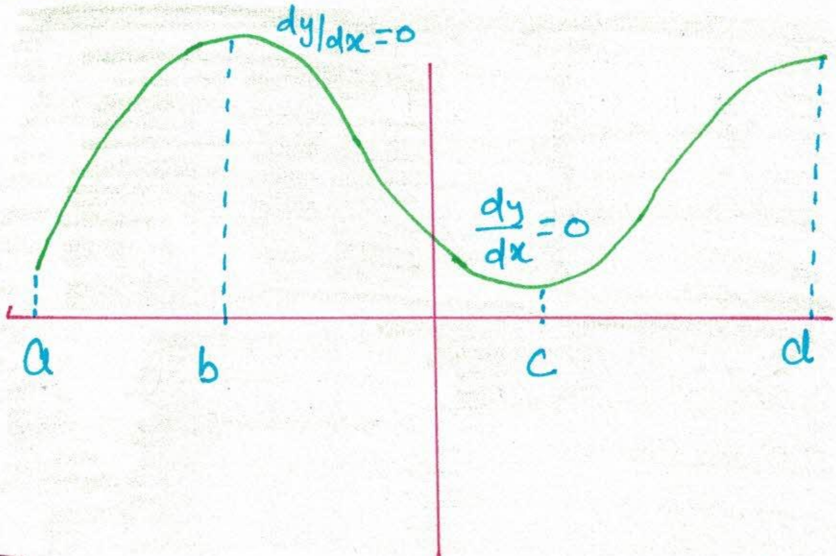
$$a \in R$$

But $\sqrt{a+4}$ is defined for $a \geq -4$

$a \in [-4, \infty) \cap (1, \infty) \Rightarrow (1, \infty)$

$$\boxed{a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)}$$

INTERVAL OF MONOTONICITY



$[a, b] \to$ Monotonic $\uparrow$

$[b, c] \to$ Monotonic $\downarrow$

$[c, d] \to$ Monotonic $\uparrow$

Question Find interval of monotonicity.

$$f(x) = \frac{\ln x}{x} \qquad \text{Domain} \quad x > 0$$

$$f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{x \times \frac{1}{x} - \ln x}{x^2}$$

$$f'(x) = \frac{1 - \ln x}{x^2}$$

$x < e$, $f'(x) > 0$

$x > e$, $f'(x) < 0$

$$\boxed{\begin{array}{l} x \in [0, e] \uparrow \\ x \in [e, \infty] \downarrow \end{array}}$$

Question

$f(x) = x^2 e^{-x^2/a^2}$, $a > 0$

$$f'(x) = x^2 \times e^{-x^2/a^2} \times \frac{-2x}{a^2} + e^{-x^2/a^2} \times 2x$$

$$f'(x) = \frac{-2x^3}{a^2} e^{-\frac{x^2}{a^2}} + 2x e^{-x^2/a^2}$$

$$= 2x e^{-x^2/a^2}\left(-\frac{x^2}{a^2} + 1\right)$$

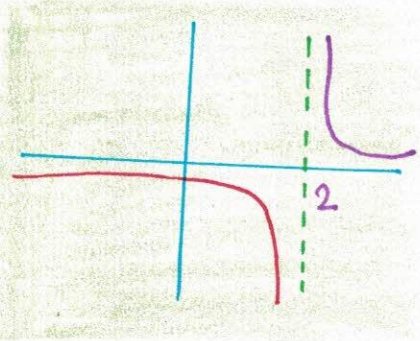
$$f'(x) \geq 0 \rightarrow 2x e^{-x^2/a^2}\left(1 - \frac{x^2}{a^2}\right) > 0$$

$$\boxed{\begin{array}{l} x \in [-\infty, -a] \cup [0, a] \uparrow \\ x \in [-a, 0] \cup [0, a] \downarrow \end{array}}$$

Question

$$f(x) = \frac{1}{x-2}$$

$$f'(x) = \frac{-1(1)}{(x-2)^2} = \frac{-1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$



$x \in [2, \infty] \downarrow$

$x \in [-\infty, 2] \downarrow$

$R \{2\}$

Question

If $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$ and $f''(x) \geq 0 \; \forall x \in R$

Then find interval of monotonicity of $g(x)$.

$\rightarrow$ $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$$= 2f'\left(\frac{x^2}{2}\right)\times\frac{2x}{2} + f'(6-x^2)\times -2x$$

$$= 2x f'\left(\frac{x^2}{2}\right) - 2x f'(6-x^2)$$

$$= 2x\left(f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right)$$

$\left\{ \text{If } f''(x)>0,\ f'(x)>0 \rightarrow f(x) \text{ increasing function} \right\}$

$$2x\left(f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right) > 0$$

Monotonicity interval is defined at critical pts.

$g'(x) = 0$ at $x=0$ and at $\frac{x^2}{2} = 6-x^2$

$$x^2 = 12 - 2x^2$$

$$3x^2 = 12 \Rightarrow x^2 = 4$$

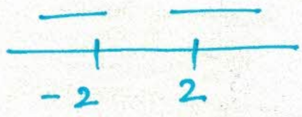
$$\boxed{x = \pm 2}$$

CASE I

$$\frac{x^2}{2} > 6-x^2$$

$f'\left(\frac{x^2}{2}\right) > f'(6-x^2)$ $\{f'(x) \uparrow \text{ function}\}$

$x^2 > 12-2x^2 \Rightarrow x^2 > 4$



$2x\left[f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right]$

$\downarrow$ $\ominus$ $\oplus$ for $x \in (-\infty,-2)$

$\downarrow$ $\oplus$ $\oplus$ for $x \in (2,\infty)$

CASE II

$$\frac{x^2}{2} < 6-x^2$$

$$x^2 < 12-2x^2$$

$3x^2 < 12 \Rightarrow x^2 < 4$, $x \in (-2,2)$

$$2x\left[f'\left(\frac{x^2}{2}\right) - f'(b-x^2)\right]$$

$$\downarrow \qquad \downarrow$$

$$\ominus \qquad \ominus$$

$x \in (-2, 0) \longrightarrow \oplus$

$x \in (0, 2) \longrightarrow \ominus$

$$\boxed{\begin{array}{ll} x \in [-\infty, -2] \cup [0, 2] & \text{decreasing} \\ x \in [-2, 0] \cup [2, \infty] & \text{increasing} \end{array}}$$

Question

If $x^2 f'(x) + 2x f(x) - x + 1 = 0$ $(x \neq 0)$ and $f(1) = 0$ then find interval of monotonicity of $f(x)$.

$$x^2 f'(x) + 2x\, f(x) - x + 1 = 0$$

$$\frac{d}{dx}\left[f(x)\,x^2\right] - x + 1 = 0$$

$$\frac{d}{dx}\left[f(x)\,x^2\right] = x - 1$$

$$\int d\left(f(x)\,x^2\right) = \int (x-1)\,dx$$

$$f(x)x^2 = \frac{x^2}{2} - x + \frac{c}{x^2}$$

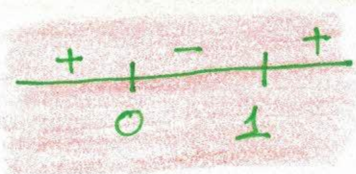
$$f(x) = \frac{1}{2} - \frac{1}{x} + \frac{c}{x^2}$$

$$f(1) = \frac{1}{2} - 1 + c = 0 \quad \Rightarrow \quad c = \frac{1}{2}$$

$$f(x) = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2} \quad \Rightarrow f'(x) = +\frac{1}{x^2} - \frac{1}{2} \times \frac{2}{x^3}$$

$$= \frac{x-1}{x^3}$$

$$f'(x) = \left(\frac{x-1}{x^3}\right)$$



$x<0,\ f(x)\uparrow$

$x \in (0,1)\ f(x)\downarrow$

$x \in (1,\infty)\ f'(x)\uparrow$

Question

Let $f'(\sin x)<0$ and $f''(\sin x)>0\ \forall\, x \in \left(0,\frac{\pi}{2}\right)$ if $g(x) = f(\sin x) + f(\cos x)$ then find interval of monotonicity of $g(x)$ in interval of $\left(0,\frac{\pi}{2}\right)$.

$\rightarrow$

$$g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$$

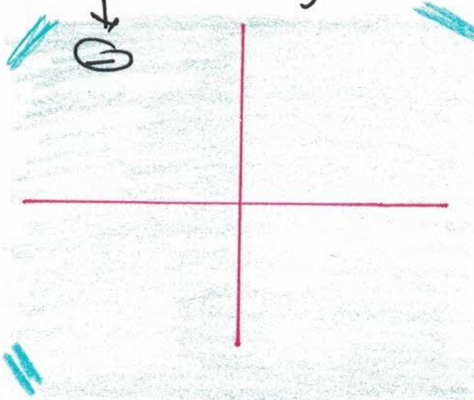
$$g'(x) = 0 \text{ at } x = \frac{\pi}{4}$$

$$g''(x) = f''(\sin x)\cos^2 x - f'(\sin x)\sin x \ldots\ldots \{f''\cos x - \sin x + f'(\cos x)\cos x\}$$

$$f'(\sin x)<0 \Rightarrow \{\underbrace{f''(\sin x)}_{\oplus}\cos^2 x + \underbrace{f''(\cos x)}_{\oplus}\sin^2 x\} - \{\underbrace{f'(\sin x)}_{\ominus}\sin x + \underbrace{f'(\cos x)}_{\ominus}\cos x\}$$

$$f'(\cos x)<0 \Rightarrow 0<x<\frac{\pi}{2}$$

$$g''(x)>0 \text{ in } \left(0,\frac{\pi}{2}\right)$$



$x \in \left(0,\frac{\pi}{4}\right)$

$\sin x < \cos x$

$f'(\sin x) < f'(\cos x) \Rightarrow g'(x)<0$

$x \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$ $\quad \sin x > \cos x \quad f'(\sin x) > f'(\cos x)$

$g'(x)>0$

$g(x)\downarrow$ in $\left(0,\frac{\pi}{4}\right)$ & $\uparrow$ in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

APPLICATION OF MONOTONICITY

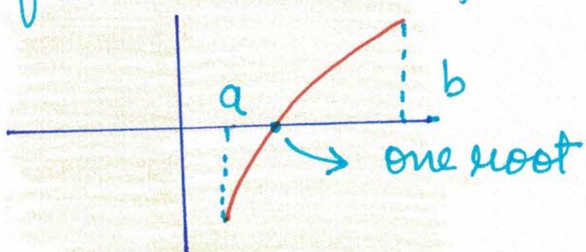
Isolation of roots of $f(x)=0$

$f(x)$ is continuous in $[a,b]$ and diff^n in $[a,b]$

$f(a)$ & $f(b)$ are of opposite sign.

$f(x)$ is either increasing or decreasing

Then $f(x)$ has Exactly one root



Ex :- P.T. $f(x) = x^3+2x^2+x+5$ has Exactly one root

Then find $[\alpha]$

$f(x) = x^3+2x^2+x+5$

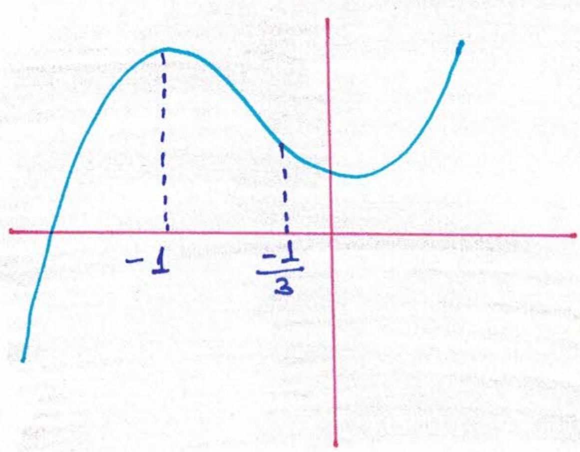
All polynomials are continuous & differentiable

$f'(x) = 3x^2+4x+1$

$= (3x+1)(x+1)$

+ | − | +

$-1 \qquad -\frac{1}{3}$



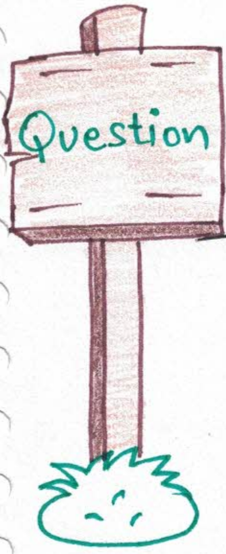
$f\left(-\frac{1}{3}\right) > 0$

$f(-1) > 0$

$f(-2) = -8+8-2+5 > 0$

$f(-3) = -27+18-3+5 < 0$

Root lies b/w (-2) & (-3) $\Rightarrow$ $\boxed{[\alpha] = -3}$



If $x - \sin x = a$ has exactly one real root in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Then find a.

$f(x) = \underbrace{x - \sin x}_{\text{cont. + diff}^n} - a$ $\qquad f'(x) = 1 - \cos x \geq 0 \downarrow$ Strictly $\uparrow$.

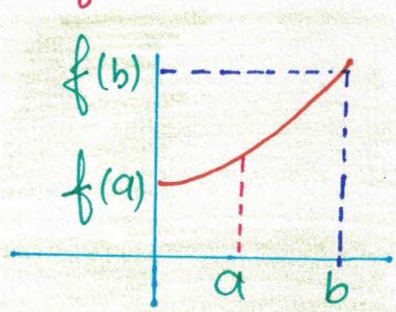
$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 1 - a \leq 0$ $\qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 - a \geq 0$

$a \geq 1 - \frac{\pi}{2}$ $\qquad a \leq \frac{\pi}{2} - 1$

$$\boxed{a \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]}$$

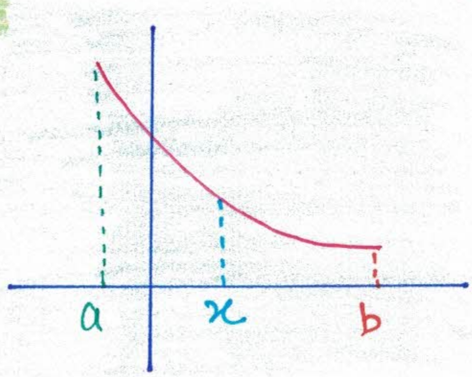
WRONG INEQUALITY USING MONOTONICITY

If $f(x)$ is continuous and strictly $\uparrow$ in $[a, b]$ then range of this function $[f(a), f(b)]$ is $f(a) \leq f(x) \leq f(b) \ \forall \ x \in [a, b]$



If $f(x)$ is continuous and strictly $\downarrow$ in $[a, b]$ then range of $f(x)$ is $[f(b), f(a)]$ i.e $f(b) \leq f(x) \leq f(a)$ $\forall \ x \in [a, b]$.

Ex:-



If $[f(x)]$ is continous and non monotonic in $[a, b]$ then least and greatest value of $f(x)$ may be at the pts where $f'(x) = 0$ or where $f'(x)$ does not exist or at end pts of function.

Question

for $x \in \left(0, \frac{\pi}{2}\right)$ P.T. $0 < x \sin x - \frac{\sin^2 x}{2} < \left(\frac{\pi - 1}{2}\right)$

$\rightarrow$ $f(x) = x \sin x - \frac{\sin^2 x}{2}$

$f'(x) = x \cos x + \sin x - \frac{\sin 2x}{2}$.

$x \cos x + \sin x - \frac{\sin 2x}{2} = 0$

$2x \cos x + 2 \sin x = \sin 2x$

$x \cos x + \sin x = \sin x \cos x$

$\Rightarrow \sin x + \cos x\,(x - \sin x)$

$\oplus$ $\oplus$

Strictly

$g(x) \Rightarrow g'(x) = 1 - \cos x$

$\oplus$

At end pts,

$x = 0$, $f(x) = 0$

$x = \frac{\pi}{2}$, $f(x) = \left(\frac{\pi}{2} - \frac{1}{2}\right)$

$$\boxed{0 < f(x) < \left(\frac{\pi - 1}{2}\right)}$$

Question

If $\frac{1}{6} < x < \frac{5}{6}$ P.T. $\frac{1}{2} < 3\left(x + \frac{1}{2\pi} - \frac{\sin \pi x}{\pi}\right) < \frac{5}{2}$

$\rightarrow$ At end pts,

$x = \frac{1}{6}$, $3\left(\frac{1}{6} + \frac{1}{2\pi} - \frac{1 \sin}{\pi} \frac{\pi}{6}\right) = 3\left(\frac{1}{6} + \frac{1}{2\pi} - \frac{1}{2\pi}\right)$

$= \frac{1}{2}$

$x = \frac{5}{6}$, $3\left(\frac{5}{6} + \frac{1}{2\pi} - \frac{1}{A} \sin \frac{5\pi}{2}\right)$

$= 3\left(\frac{5}{6} + \frac{1}{2\pi} - \frac{1}{2\pi}\right) = \frac{5}{2}$

$$\boxed{\left\{0 < f(x) < \frac{5}{2}\right\}}$$

H.P

$f(x) = 3\left(x + \frac{1}{2\pi} - \frac{\sin \pi x}{\pi}\right)$

$f'(x) = 3\left(1 + 0 - \frac{1}{\pi} \cos \pi x \times \pi\right)$

$= 3(1 - \cos \pi x) = \oplus > 0$

Strictly ↑ Range is $(f(a), f(b))$.

GENERAL APPROCH OF PROVING INEQUALITY USING MONO.

Suppose we have to prove $f(x) \geq g(x)\ \forall\ x \geq a$

Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(a)$

Let $h'(x) > 0$

$h(x)$ is strictly $\uparrow\ \forall\ x \geq a$

$x \geq a$

$h(x) \geq h(a) \Rightarrow$ if $h(a) = 0$

$h(x) \geq 0$

$f(x) - g(x) \geq 0 \Rightarrow \boxed{f(x) \geq g(x)}$

Question

P.T. $\sin x < x < \tan x\ \forall\ x \in (0, \frac{\pi}{2})$

$\rightarrow$ Let $g(x) = x - \sin x$

$g'(x) = 1 - \cos x$

$g'(x) > 0$

$g(x)$ is strictly $\uparrow$

$g(x) > g(0)$

$x \sin x > 0$

$x > \sin x$

Let $h(x) = x - \tan x$.

$h'(x) = 1 - \sec^2 x$

$h'(x) < 0$

$h(x)$ is strictly $\downarrow$

$h(x) < h(0)$

$x - \tan x < 0$

$x < \tan x$

$$\boxed{\sin x < x < \tan x}$$

Question

find ordered relation b/w x and $\tan^{-1} x$.

Let $h(x) = x - \tan^{-1} x$

$$h'(x) = 1 - \frac{1}{1+x^2} > 0 \left\{ \frac{1+x^2-1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0 \right\}$$

$h(x)$ is strictly $\uparrow$ for all $x \in R$.

$x > 0$

$h(x) > h(0)$

$x - \tan^{-1}x > 0$

$x \tan^{-1} x.$

$0 > x$

$h(0) > h(x)$

$0 > x - \tan^{-1} x$

$\tan^{-1} x > x.$

$$x > \tan^{-1} x \ (x \in (0,\infty))$$

$$\tan^{-1} x > x \ (x \in (-\infty, 0))$$

Question → Find the relation b/w $(2014)^{\frac{1}{2014}}$ & $(2013)^{\frac{1}{2013}}$

$$f(x) = x^{\frac{1}{x}}.$$

$$\log f(x) = \frac{1}{x} \log x.$$

$$\frac{1}{f(x)} f'(x) = \frac{1}{x} \times \frac{1}{x} - \log x \times \frac{1}{x^2}$$

$$f'(x) = x^{\frac{1}{x}} \left(\frac{1}{x^2} - \frac{\log x}{x^2} \right)$$

$$= \frac{x^{1/x}(1 - \log x)}{x^2}$$

$f'(x) > 0, \quad x < e \quad x \in (0, e)$

$f'(x) < 0, \quad x > e \quad x \in (e, \infty)$

$f(x) \uparrow, \ x \in (0, e)$

$f(x) \downarrow, \ x \in (e, \infty)$

$$\boxed{2013^{\frac{1}{2013}} > 2014^{\frac{1}{2014}}}$$

Question find relation b/w π^e & e^π

$f(x) = x^{1/x}$ is $\downarrow$

$\pi > e.$

$e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$

Take e^π power

$e^{\frac{1}{e} \times e\pi} > \pi^{\frac{1}{\pi} \times e\pi} \Rightarrow \boxed{e^\pi > \pi^e}$

Question P.T. $\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{1+e^2}} < \left(\tan^{-1}e\right)^2 + \frac{2}{\sqrt{1+e^2}}$

$\longrightarrow$ we have to prove $f\left(\frac{1}{e}\right) < f(e)$

$f\left(\frac{1}{x}\right) < f(x) \quad e > \frac{1}{e}$

$f(x)$ is strictly $\uparrow$.

$$f(x) = \left(\tan^{-1}x\right)^2 + \frac{2x}{\sqrt{1+x^2}}$$

$$f'(x) = \frac{2\tan^{-1}x}{(1+x^2)} + 2\left[\frac{\sqrt{1+x^2} - \frac{2x^2}{2\sqrt{1+x^2}}}{1+x^2}\right]$$

$$= \frac{2}{1+x^2}\left[\tan^{-1}x - \frac{2}{\sqrt{1+x^2}}\right]$$

Let $\tan^{-1}x - \frac{2}{\sqrt{1+x^2}} = g(x)$

$$g'(x) = \frac{\frac{1}{1+x^2} - \left\{-2 \times \frac{1}{2\sqrt{1+x^2}}x\right\}}{1+x^2}$$

$$= \frac{\frac{1}{1+x^2} + \frac{x}{\sqrt{1+x^2}}}{1+x^2} \longrightarrow (+)$$

$g'(x) > 0$

$g(x) \uparrow$ strictly

$f(x)$ is strictly $\uparrow$.

$f(x) > f\left(\frac{1}{x}\right)$

$$\boxed{f(e) > f\left(\frac{1}{e}\right) \quad \text{H.P.}}$$

Question

If $ax^2 + \frac{b}{x} \geq c \ \forall x \in R^{\oplus}$. a, b, c are +ve constant then P.T. $27\,ab^2 \geq 4c^3$.

$\rightarrow$ Let $f(x) = ax^2 + \frac{b}{x} - c$

$f'(x) = 2ax - \frac{b}{x^2} = 0$

$x = \left(\frac{b}{2a}\right)^{\frac{1}{3}} \longrightarrow R^{\oplus}$

$a \times \left(\frac{b}{2a}\right)^{\frac{2}{3}} + \frac{b}{\left(\frac{b}{2a}\right)^{\frac{1}{3}}} \geq c$

$a^3 \times \left(\frac{b}{2a}\right)^2 + \left[b\left(\frac{2a}{b}\right)^{\frac{1}{3}}\right]^3 + 3ab\left(\frac{b^2}{4a^2} \times \frac{2a}{b}\right)^{\frac{1}{3}}$

$\left[a\left(\frac{b}{2a}\right)^{\frac{1}{3}} + b \times \left(\frac{2a}{b}\right)^{\frac{1}{3}}\right]$

$a^3 \times \frac{b^2}{4a^2} + b^3 \times \frac{2a}{b} + \frac{3b^2}{2}\left[a\left(\frac{b}{2a}\right)^{\frac{2}{3}} + b\left(\frac{2a}{b}\right)^{\frac{1}{3}}\right] \geq c^3$

Solve, we will get.

$$\boxed{27\,ab^2 \geq 4c^3}$$

Question

If $x + ax^{-2} > 2 \ \forall \ x \in R^+$, then find least +ve natural no. a.

$$\longrightarrow \frac{\frac{x}{2} + \frac{x}{2} + \frac{a}{x^2}}{3} > \left(\frac{a}{4}\right)^{\frac{1}{3}}$$

$$x + \frac{a}{x^2} > 3\left(\frac{a}{4}\right)^{\frac{1}{3}} \qquad (2)^3 > (3)^3\left(\frac{a}{4}\right) \quad a < \frac{3^2}{27}$$

$$a = 1$$

$$3\left(\frac{a}{4}\right)^{\frac{1}{3}} > 2 \Rightarrow \frac{a}{4} > \frac{8}{27} \Rightarrow a > \frac{32}{27}.$$

Least +ve natural no. $\Rightarrow \boxed{a = 2}$

Question

Find the least +ve value of 'a' for which $\ln x \leq ax^2 \ \forall \ x \in R^+$.

$$f(x) = ax^2 - \ln x$$

$$f'(x) = 2ax - \frac{1}{x} = 0$$

$$2ax = \frac{1}{x} \Rightarrow x^2 = \frac{1}{2a} \Rightarrow x = +\frac{1}{\sqrt{2a}}$$

$$a \times \frac{1}{2a} - \ln\left(\frac{1}{\sqrt{2a}}\right) \geq 0$$

$$\ln\left(\frac{1}{\sqrt{2a}}\right) \leq \frac{1}{2}$$

$$\frac{1}{\sqrt{2a}} \leq e^{\frac{1}{2}} \Rightarrow \sqrt{a} \geq \frac{1}{\sqrt{2}e^{\frac{1}{2}}}$$

$$\boxed{a \geq \frac{1}{2e} \Rightarrow a = \frac{1}{2}}$$

ALGEBRA OF MONOTONICITY

① **NEGATIVE** Let $f(x)$ be a strictly ↑ function for $[a, b]$ then $-f(x)$ will be strictly ↓ for $[a, b]$.

EX:- $\tan^{-1}x > 0$ for $\forall\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ↑

$-\tan^{-1}x < 0$ for $\forall\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ↓

② **RECIPROCAL** Let $f(x)$ be a non zero strictly ↑ function for $[a, b]$ then $\frac{1}{f(x)}$ will be strictly ↓ for $[a, b]$ i.e. $\frac{1}{I} = D$, $\frac{1}{D} = I$

③ **DIFFERENCE** Let $h(x) = f(x) - g(x)$

$$h(x) = \underset{\substack{\downarrow \\ I}}{f(x)} + \underset{\substack{\downarrow \\ D}}{(-g(x))}$$

$h(x) = I + D$

$h(x) \longrightarrow$ no comment $= I + D$

If $f(x) = D$, $g(x) = I$

$$f(x) - g(x) = D - I = D + (-I)$$
$$= D + D$$
$$= D$$

Therefore

$I - D = I + (-D) = I + I \longrightarrow$ Increasing

$I - I = I + (-I) = I + D \longrightarrow$ no comment

$D - I = D + (-I) = D + D \longrightarrow$ decreasing

4 PRODUCT Let $f(x)$ + $g(x)$ be two +ve value strictly ↑ function, then $h(x) = f(x) \cdot g(x)$

$$h'(x) = \underset{+}{f(x)}\,\underset{>0}{g'(x)} + \underset{>0}{f'(x)}\,\underset{+}{g(x)}$$

$h'(x) > 0 \longrightarrow$ Strictly ↑.

Now, let $f(x)$ + $g(x)$ be strictly function.

$$h(x) = f(x)\, g(x)$$

$$h'(x) = \underset{-}{f'(x)}\,\underset{+}{g(x)} + \underset{-}{g'(x)}\,\underset{+}{f(x)}$$

$h'(x) < 0 \longrightarrow$ Strictly ↓.

☆ Let $f(x)$ be strictly ↑ with −ve value.
$g(x)$ be strictly ↓ with +ve value.

$$h(x) = f(x)\, g(x)$$

$$h'(x) = \underset{+}{f'(x)}\,\underset{+}{g(x)} + \underset{-}{g'(x)}\,\underset{-}{f(x)}$$

$h'(x) > 0 \longrightarrow$ Strictly ↑

EX:- $f(x) = \ln x$, $x \in (0,1)$ strictly ↑ with −ve value

$g(x) = \cot^{-1} x$, strictly ↓ with +ve value.

$$h(x) = \ln x \cot^{-1} x$$

$$h'(x) = \frac{-\overset{-}{\ln x}}{1+x^2} + \frac{\cot^{-1}x \rightarrow +}{x \rightarrow +}$$

$h'(x) > 0 \longrightarrow$ Strictly ↑

COMPOSITION OF MONOTONICITY

Let $f(x)$ be strictly $\uparrow$ in $[a, b]$ and $g(x)$ be strictly $\uparrow$ in $[f(a), f(b)]$, then $g(x)$ is also strictly $\uparrow$ in $[a, b]$.

So, composition of $\boxed{I(I) = I}$

Let $f(x)$ be strictly $\downarrow$ in $[a, b]$ and $g(x)$ be strictly $\downarrow$ in $[f(b), f(a)]$, then $gof(x)$ is is strictly $\uparrow$ in $[a, b]$.

So, composition of $\boxed{D(D) = I}$

EX:- $h(x) = \log_{\frac{1}{2}} x \downarrow$

$g(x) = \cot^{-1} x \downarrow$

$f(x) = gof(x) \Rightarrow \cot^{-1}(\log_{\frac{1}{2}} x) \rightarrow$ Strictly $\uparrow$

$gof(x) = h(x)$

$h'(x) = \underset{\ominus}{g'f(x)}\ \underset{\ominus}{f'(x)} > 0$

$h(x)$ is strictly $\uparrow$

So, if $f(x)$ be strictly $\uparrow$ in $[a, b]$ and $g(x)$ be strictly $\downarrow$ in $[f(a), f(b)]$, then $gof(x)$ is strictly $\downarrow$ in $[a, b]$.

$$g \circ f(x) = h(x)$$

$$h'(x) = \underbrace{g' \circ f(x)}_{\ominus} \; \underbrace{f'(x)}_{\oplus} < 0$$

$h(x)$ is strictly $\downarrow$

$I(I) = I \qquad I(D) = D$

$D(D) = I \qquad D(I) = D$

Question

Find the interval of monotonicity of -

i) $f(x) = e^{\sin x + \cos x}$

$\rightarrow g(x) = e^x \uparrow \qquad h(x) = \sin x + \cos x$

$\qquad\qquad\qquad\qquad h'(x) = \cos x - \sin x$

$$h'(x) = \sqrt{2} \sin\left(\frac{\pi}{4} - x\right) > 0$$

$$\frac{\pi}{4} - x \in (0, \pi)$$

$$-x \in \left(\frac{-\pi}{4}, \frac{3\pi}{4}\right)$$

$$x \in \left(\frac{3\pi}{4}, \frac{\pi}{4}\right)$$

$g \circ h(x) \uparrow$ in $\left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$

$g \circ h(x) \downarrow$ in others.

ii) $f(x) = \sqrt{2x - x^2} \qquad 2x - x^2 \geq 0 \quad x^2 < 2x \quad x \leq 2$

$h(x) = \sqrt{x} \uparrow \qquad g(x) = 2x - x^2$

$x \geq 0 \qquad g'(x) = 2 - 2x$

$2-2x > 0$

$2 > 2x \quad x < 1$

$x \in [0,1] \quad \log(x) = f(x) \uparrow$

$2-2x < 0 \Rightarrow x > 1$

$x \in [1,2] \quad \log(x) = f(x) \downarrow$

Question

$f(x) = \tan^{-1}(\sin x + \cos x)^3$

$\rightarrow h(x) = \tan^{-1}(x^3) \uparrow. \qquad g(x) = (\sin x + \cos x)$

$g'(x) = \cos x - \sin x$

$= \sqrt{2} \sin\left(\frac{\pi}{4}\right)$

$-x + \frac{\pi}{4} \in [0, \pi]$

$x \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

$h(x) \downarrow$ in other

$h(x) \uparrow$ in $\left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$

Question

$f(x) = \sin(\ln x) - \cos(\ln x)$

$f(x) = \sqrt{2} \sin\left(\ln x - \frac{\pi}{4}\right)$

$f'(x) = \frac{\sqrt{2}}{x} \cos\left(\ln x - \frac{\pi}{4}\right)$

$\ln x - \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\ln x \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$

For $x \in \left[e^{2n\pi - \frac{\pi}{4}}, e^{2n\pi + \frac{3\pi}{4}}\right]$, $f(a) \uparrow$

$\ln x - \frac{\pi}{4} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$\ln x \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

$x \in \left[e^{2n\pi + \frac{3\pi}{4}}, e^{2n\pi + \frac{5\pi}{4}}\right]$, $f(x) \downarrow$

Question

Find the ordered relation b/w $\sin(\cos x)$ and $\cos(\sin x)$ for $x \in (0, \frac{\pi}{2})$.

$f(x) = x - \sin x$

$f'(x) = 1 - \cos x > 0$

$x > 0$

$f(x) > f(0)$

$x - \sin x > 0$

$x > \sin x$

For $x \in (0, \frac{\pi}{2})$

$\cos \in (0, 1) \rightarrow$ Sub interval

$\cos x > \sin(\cos x) \Rightarrow x > \sin x$

$\cos x < \cos(\sin x) \longrightarrow \cos x \downarrow$ function

$\cos(\sin x) > (\cos x) > \sin(\cos x)$

$$\boxed{\cos(\sin x) > \sin(\cos x)}$$

Question

P.T. $\sin^2 x < x \sin(\sin x) \; \forall \; x \in (0, \pi/2)$.

$\longrightarrow \frac{\sin x}{x} < \frac{\sin(\sin x)}{\sin x}$

Let $f(x) = \dfrac{\sin x}{x}$

$f'(x) = \dfrac{x\cos x - \sin x}{x^2}$

Let $l(x) = x\cos x - \sin x$

$l'(x) = -x\sin x + \cos x - \cos x$

$= -x\sin x < 0$

$l(x) \downarrow$ function

$f(x) \downarrow$ function

So we have to Prove that $x > \sin x$

let $g(x) = x - \sin x$

$g'(x) = 1 - \cos x > 0$

$g'(x) \uparrow$ function

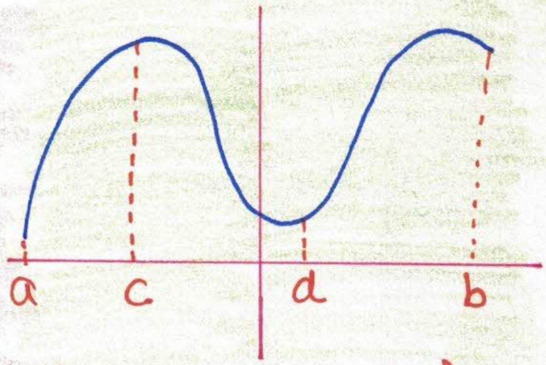
$x > 0$

$g(x) > g(0) \Rightarrow x - \sin x > 0$

$$\boxed{x > \sin x}$$

CRITICAL POINTS

If $f(x)$ is defined for $[a,b]$ and let c be any interior pt. of $f(x)$ then if $f'(c) = 0$ or $f'(c)$ does not exist, then c is called critical pt. of $f(x)$.



{ c must be in domain }

Stationary Point is a point on the graph of $f(x)$ where $f'(x) = 0$ only within its domain

Question

Find the critical pts. of :-

$$f(x) = \frac{e^x}{x-2}$$

Domain $x \neq 2$

$$f'(x) = \frac{(x-2)e^x - e^x}{(x-2)^2} = \frac{xe^x - 3e^x}{(x-2)^2} = \left(\frac{x-3}{x-2}\right)e^x$$

$f'(x) = 0$ at $x = 3$

$x=3$ is critical pt.

Question

$$f(x) = (x-2)^{2/3}$$

$$f'(x) = \frac{2}{3}(x-2)^{-1/3} = \frac{2}{3\sqrt[3]{(x-2)}}$$

$$\frac{2}{3\sqrt[3]{x-2}} = 0$$

At $x = 2$ $f'(x)$ does not exist

$x = 2$

$f(x) = x^3 - 3x + 4$

$f'(x) = 3x^2 - 3$

$x = \pm 1$ are critical pts.

Question

If $f(x) = (2x-3)\sin x - (a^2-4)x + 2$ has no critical pts, then find a.

$\rightarrow f'(x) = (2x-3)\cos x + 2\sin x - (a^2-4) \neq 0$

$(2a-3)\cos x - (a^2+2) \neq 0$

$(2a-3)\cos x \neq a^2+2$

$\cos x \neq \dfrac{a^2+2}{2a-3}$

$\dfrac{a^2+2}{2a-3} > 1$

$\dfrac{a^2+2}{2a-3} - 1 > 0$

$\dfrac{a^2+2-2a+3}{2a-3} > 0$

$\dfrac{a^2-2a+5}{2a-3} > 0$

$\dfrac{a^2+2}{2a-3} < -1$

$\dfrac{a^2+2}{2a-3} + 1 < 0$

$\dfrac{a^2+2a-1}{2a-3} < 0$

$a \in \left(+\frac{3}{2}, \infty\right)$

Question

If $f(x) = (a^2-3a+2)(\cos^2 x - \sin^2 x) - (a-2)x + 2014$ has no critical pt, Find a.

$\rightarrow f'(x) = (a^2-3a+2)(-\sin 2x - \sin 2x) - (a-2)$

$f'(x) = -2(a^2-3a+2)\sin 2x - (a-2)$

$-2(a-2)(a-1)\sin 2x - (a-2)$

$(a-2)\{-2(a-1)\sin 2x-1\}\neq 0$

$a\neq 2$ $\quad -2(a-1)\sin 2x-1\neq 0$

$(a-1)\sin 2x\neq \frac{-1}{2}$

$\sin 2x\neq \frac{-1}{2(a-1)}$ $\quad \{2a-2<1\}$

$\frac{-1}{2(a-1)}>1$

$-1>2a-2$

$2a<1$

$a<\frac{1}{2}$

$\frac{-1}{2(a-1)}<-1$

$-\frac{1}{2a-2}+1<0$

$\frac{-1+2a-2}{2(a-1)}<0$

$\frac{2a-3}{2(a-1)}<0$ $\quad$ + 1 − 3/2 +

Question If $f(x)=e^{2x}-(a+1)e^{x}+2x$ strictly $\uparrow\ \forall x\in R$. Then find a.

$\longrightarrow f'(x)=2e^{2x}-(a+1)e^{x}+2\geq 0$

$e^{x}=t \Rightarrow 2t^{2}-(a+1)t+2\geq 0$

$D\leq 0$ $\quad$ + −5 − 3 +

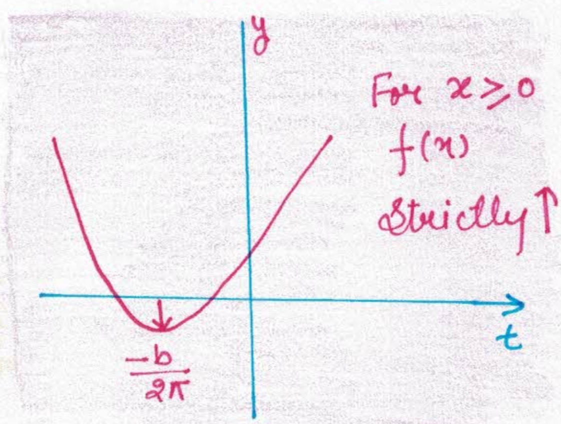
CASE-I $\quad (a+1)^2-8(2)\leq 0$

$a^2+1+2a-16\leq 0$

$a^2+2a-15\leq 0$

$(a+5)(a-3)\leq 0$ $\quad \frac{-b}{2a}<0$

$\boxed{a\in[-5,3]}$ $\quad \frac{a+1}{4}<0$



CASE-II $\quad f(0)>0 \quad D\geq 0$

$(a+1)^2-16\geq 0 \quad (a+5)(a-3)\geq 0$

$\boxed{a\in[-\infty,-5]\cup[3,\infty]}$

Question

If $f(x) = \sin 2x - 8(b+2)\cos x - (4b^2+16b+6)x$ is strictly $\downarrow$, find b.

$$f'(x) = 2\cos 2x + 8(b+2)\sin x - (4b^2+16b+6)$$

$$= 2(1-2\sin^2 x) + 8(b+2)\sin x - (4b^2+16b+6)$$

$$= 2 - 4\sin^2 x + 8(b+2)\sin x - (4b^2+16b+6)$$

$$= -4\sin^2 x + 8(b+2)\sin x - (4b^2+16b+4) \leq 0$$

$$\sin^2 x = 2(b+2)\sin x + (b^2+4b+1) \geq 0$$

CASE I

$f(1) > 0$

$$t^2 - 2(b+2)t + (b^2+4b+1) \geq 0$$

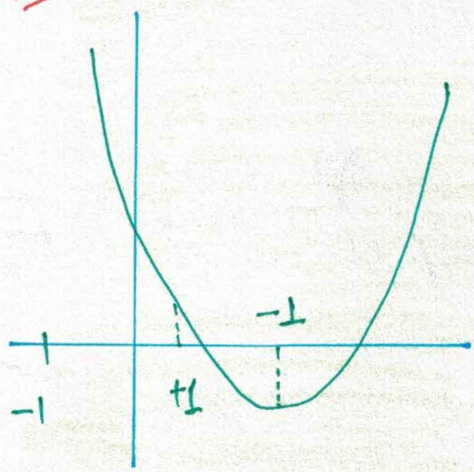
$$1 - 2(b+2) + b^2 + 4b + 1 \geq 0$$

$$1 - 2b - 4 + b^2 + 4b + 1 \geq 0$$

$$b^2 + 2b - 2 \geq 0$$

$$b = -1 \pm \sqrt{3}$$

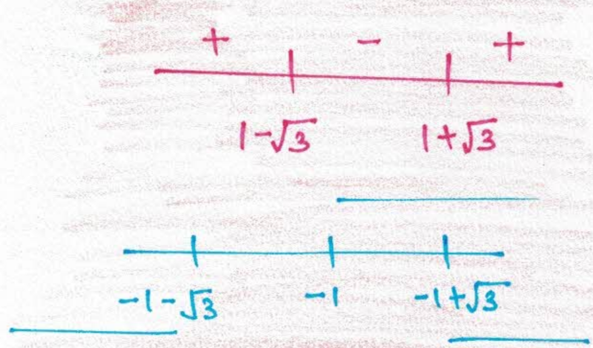
$$b \in [-\infty, -1-\sqrt{3}] \cup [-1+\sqrt{3}, \infty)$$



$$\left[\frac{-b}{2a} > 1\right] \quad \frac{+2(b+2)}{2} > 1$$

$$b+1 > 0 \qquad b > -1$$

$$b \in (-1, +\sqrt{3}, \infty)$$



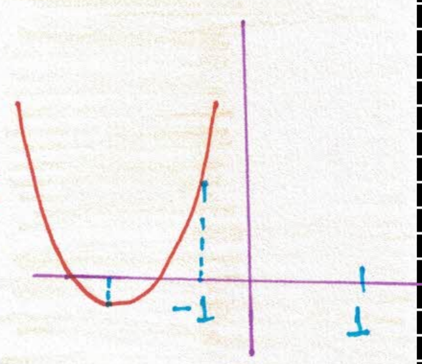
CASE II

$f(-1) > 0$

$$1 + 2b + 4 + b^2 + 4b + 1 > 0$$

$$b^2 + 6b + 6 > 0$$

$$b = \frac{-6 \pm \sqrt{36-24}}{2} = \frac{-6 \pm 2\sqrt{3}}{2} = -3 \pm \sqrt{3}$$



$$b \in (-\infty, -3-\sqrt{3}) \cup (-3+\sqrt{3}, \infty)$$

$\frac{-b}{2a} < -1 \qquad \frac{2(b+2)}{2} + 1 < 0 \Rightarrow \quad b+3 < 0$

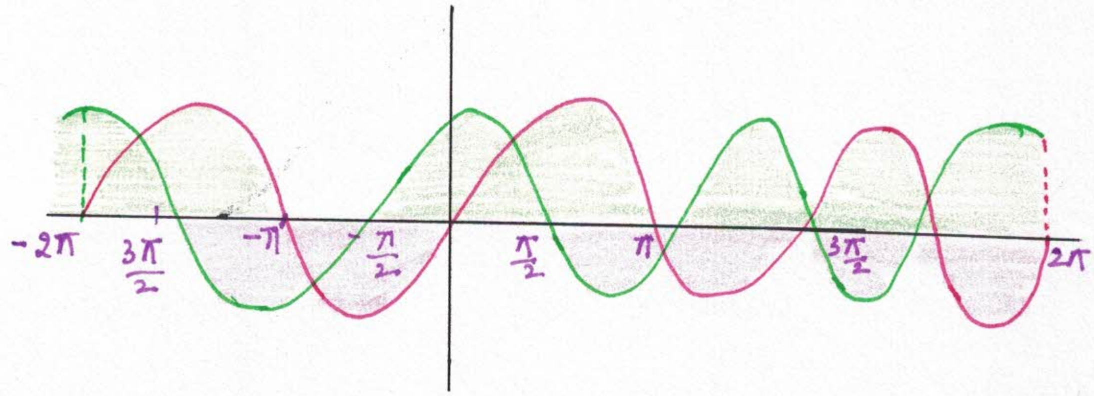
$b < -3$

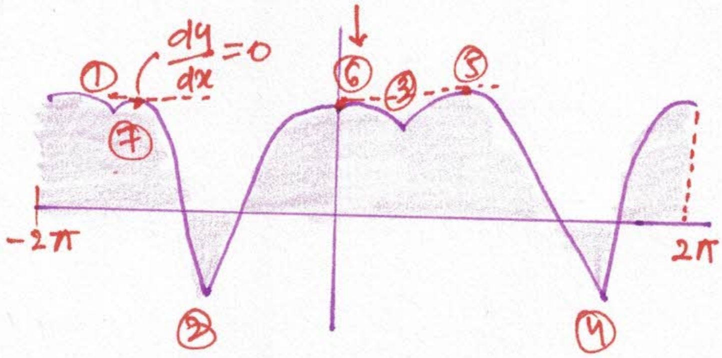
$b \in (-\infty, -3-\sqrt{3})$

$$\boxed{b \in (-\infty, -3-\sqrt{3}) \cup (-1+\sqrt{3}, \infty)}$$

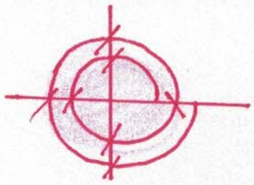
Question

find the no. of critical Points of $f(x) = \max(\sin x, \cos x)$ in $(-2\pi, 2\pi)$.





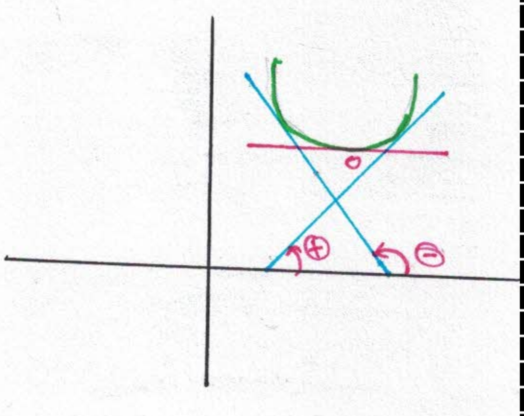
Total 7 Points



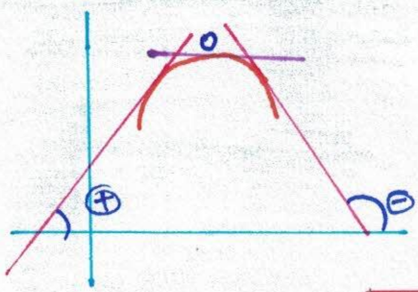
CONCAVITY & POINT OF INFLECTION

$f(x)$ is defined for $[a, b]$ slope of tangent ↑ with ↑ in value of x.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \geq 0$$



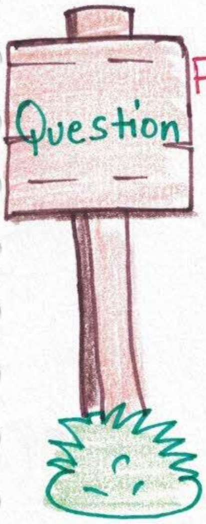
$\boxed{f''(x) \geq 0 \ \forall x \in [a, b]}$ curve is concave up.



$f(x)$ is defined for $[a,b]$ Slope of tangent ↓ with ↑ in value of x.

$$\frac{d}{dy}\left(\frac{dy}{dx}\right) \leq 0$$

$\boxed{f''(x) \leq 0}$ Curve is Concave down



Find interval of concavity of $f(x) = x + \sin x$.

→ $f'(x) = 1 + \cos x$

$f''(x) = -\sin x$

$f''(x) > 0, \; x \in [\pi, 2\pi]$

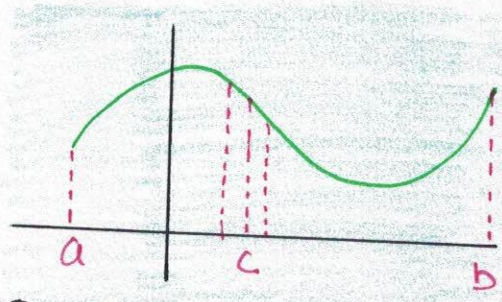
$f''(x) < 0, \; x \in [0, \pi]$

Concave up for $x \in [\pi, 2\pi]$

Concave up for $x \in [0, \pi]$

$f(x)$ is defined for $[a,b]$

$f(x)$ changes its concavity at $[c, f(c)]$ i.e. $f''(c-h) < 0, f''(c+h) > 0$

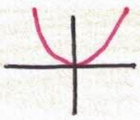


If graph of $f(x)$ changes its concavity around a pt. $x = c$, then $[c, f(c)]$ is called point of inflection.

$f''(x) = 0$ is not necessary condition, the main continuity is that at that pt. graph should change its concavity.

e.g $y = x^4$

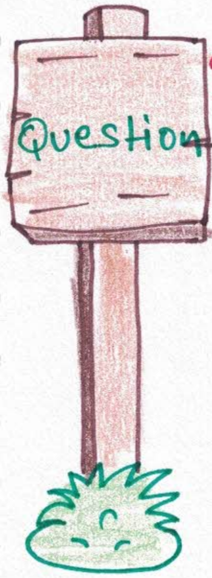
$f''(x)\Big|_{x=0} = 0$ but $f(x)$ does not changes its concavity.



$f''(c-h) > 0$ and $f''(c+h) < 0$

$f''(c-h) < 0$ and $f''(c+h) > 0$

then "c" is called Point of concavity.



Find Point of inflection :-

$$f(x) = x^2 e^{-x}$$

$$f'(x) = -x^2 e^{-x} + 2e^{-x} x.$$

$$f''(x) = -\left[2xe^{-x} - x^2 e^{-x}\right] + \left[2e^{-x} - 2xe^{-x}\right]$$

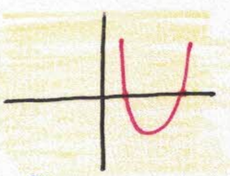
$$= -2xe^{-x} + x^2 e^{-x} + 2e^{-x} - 2xe^{-x}$$

$$f''(x) = x^2 e^{-x} - 4xe^{-x} + 2e^{-x}$$

$$f''(x) = 0 \Rightarrow x^2 e^{-x} - 4xe^{-x} + 2e^{-x} = 0$$

$$e^{x}(x^2 - 4x + 2) = 0$$

$$x = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2} = (2 \pm \sqrt{2})$$



$$e^{x}\left(x - (2+\sqrt{2})\right)\left(x - (2-\sqrt{2})\right) = 0$$

At $(2+\sqrt{2}+h)$, $f''(x) = \underset{\oplus}{e^{-x}}\ \underset{\oplus}{(x^2 - 4x + 2)} > 0$

$(2+\sqrt{2}-h)$, $f''(x) = \underset{\oplus}{e^{-x}}\ \underset{\ominus}{(x^2 - 4x + 2)} < 0$

At $2-\sqrt{2}+h$, $f''(x) = \underset{\oplus}{e^{-x}}\ \underset{\ominus}{(x^2 - 4x + 2)} < 0$

$2-\sqrt{2}-h$, $f''(x) = \underset{\oplus}{e^{-x}}\ \underset{\oplus}{(x^2 - 4x + 2)} > 0$

$\boxed{x = 2+\sqrt{2},\ 2-\sqrt{2} \text{ are points of inflection}}$

Question

$$f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{x \times \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)\, 2x}{x^4}$$

$$= \frac{-x - 2x + 2x \ln x}{x^4} = \frac{2x \ln x - 3x}{x^4}$$

$$f'''(x) = 0 \Rightarrow \frac{2x \ln x - 3x}{x^4} = 0$$

($x = 0$ not in domain)

$$= 2x \ln x - 3x = 0$$

$$2x \ln x = 3x$$

$$\ln x = \frac{3}{2} \Rightarrow x = e^{3/2}$$

$$= \frac{x(2\ln x - 3)}{x^4} = \frac{-x(3 - 2\ln x)}{x^4}$$

$e^{3/2} + h$, $f''(x) = \oplus$

$e^{3/2} - h$, $f''(x) = \ominus$

($x = e^{3/2}$ is pt. of inflection)

Question

$$f(x) = (x+2)^{1/3}$$

$$f'(x) = \frac{1}{3}(x+2)^{-\frac{2}{3}}$$

$$f''(x) = -\frac{2}{9}(x+2)^{-\frac{5}{3}} = \frac{-2}{9(x+2)^{5/3}}$$

$x = -2 + h$, $f''(x) = \ominus$

$x = -2 - h$, $f''(x) = \oplus$

$x = -2$ is pt. inflection.

Question

$$f(x) = (x-2)^{\frac{5}{3}} + 2$$

$$f'(x) = \frac{5}{3}(x-2)^{2/3}$$

$$f''(x) = \frac{10}{9}(x-2)^{\frac{-1}{3}} = \frac{10}{9(x-2)^{1/3}}$$

At $x = 2+h$, $f''(x) = \oplus$

$x = 2-h$, $f''(x) = \ominus$

$x = 2$ is a point of inflection.

Question

Consider a curve $y = \cos^{-1}(2x-1)$ and a line $2Px - 4y + 2\pi - P = 0$. If line intersects the curve at 3 distinct pts. Find possible value of P ?

$\rightarrow$ $y = \cos^{-1}(2x-1)$

$$y' = \frac{-1}{\sqrt{1-(2x-1)^2}} \times 2 = \frac{-2}{\sqrt{1-4x^2-1+4x}} = \frac{-2}{\sqrt{4x-4x^2}}$$

$$y'' = 2 \times \frac{1}{4\sqrt{x-x^2}} \times \frac{4-8x}{(4x-4x^2)} = \frac{2\times 4(1-2x)}{4\sqrt{x-x^2}\times 4(x-x^2)}$$

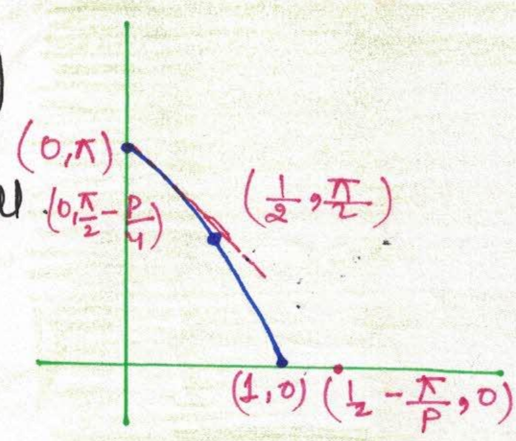
$$= \frac{1-2x}{2\sqrt{x-x^2}\,(x-x^2)}$$

$x = \frac{1}{2}$ is Pts. of inflection.

At $x = \frac{1}{2}$, $y = \cos^{-1}\left(2\times\frac{1}{2} - 1\right) = \frac{\pi}{2}$

Point of inflection $= \left(\frac{1}{2}, \frac{\pi}{2}\right)$

$\left(\frac{1}{2}, \frac{\pi}{2}\right)$ lies on the line as well



$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = \frac{P}{4}$$

$$\text{Slope of line} = \frac{\pi - \frac{\pi}{2}}{0 - \frac{1}{2}} = -\pi$$

If slope of curve > slope of line at $x=\frac{1}{2}$, then only inter-section occurs.

$$\therefore \frac{P}{2} \in [-\pi, -2] \Rightarrow \boxed{P \in [-2\pi, -4]}$$

Question

P.T. $\sin x + 2x \geq \frac{3x(x+1)}{\pi} \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$

$\rightarrow f(x) = \sin x + 2x$

$f'(x) = \cos x + 2$

$f''(x) = -\sin x$

$g(x) = \frac{3x^2 + 3x}{\pi}$

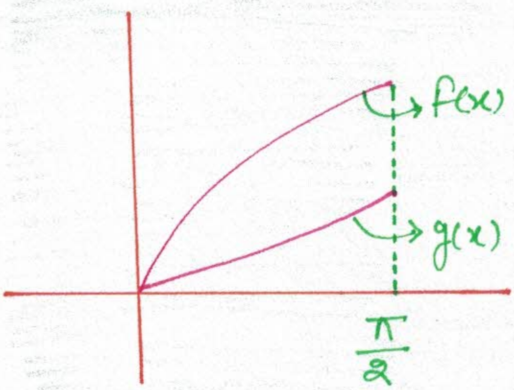
$g'(x) = \frac{\pi(6x+3)}{\pi^2}$

$= \frac{6x+3}{\pi}$

$f''(x) < 0$, for $x \in [0, \pi]$ Concave ↓

$g''(x) = \frac{\pi(6)}{\pi^2} = \frac{6}{\pi} > 0$

$f''(x) > 0$, $x \in [\pi, 2\pi]$ Concave ↑



At $\frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = \pi + 1$

$\frac{\pi}{2}$, $g\left(\frac{\pi}{2}\right) = \frac{3\pi}{4} + \frac{3}{2} = \pi - \frac{\pi}{4} + \frac{3}{2}$

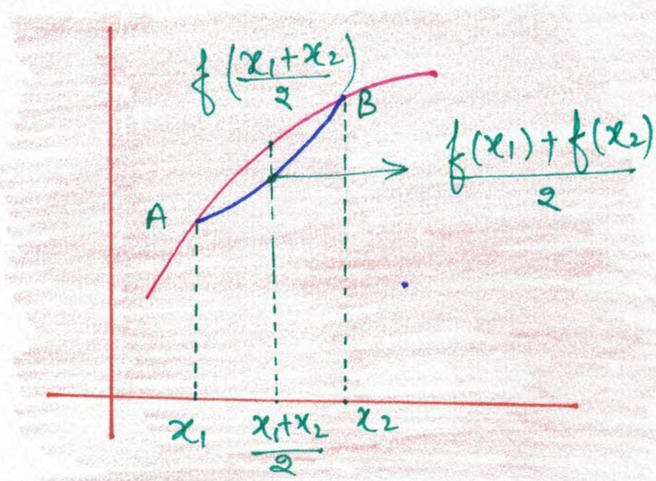
$g\left(\frac{\pi}{2}\right) < f\left(\frac{\pi}{2}\right)$

Hence proved

Question

If $f(x) > 0$ and $f''(x) < 0$ then P.T. for any $x_1, x_2 \in \mathbb{R}$ $(x_1 < x_2)$, $f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2}$

$\rightarrow f(x)$ strictly ↑ & concave downwards.



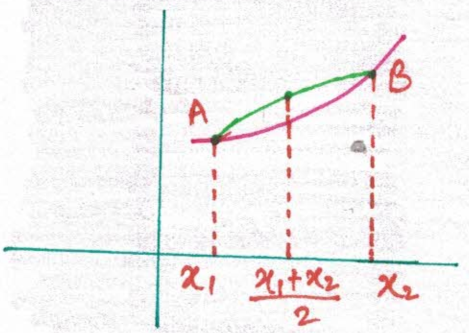
$y_{curve} > y_{m.p.}$

$$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$$

H.P.

Question If $f'(x)>0,\ f''(x)>0$. P.T. $f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$

$\rightarrow$ $f(x)$ strictly $\uparrow$ & concave upwards.



$$y_{mp} > y_c$$

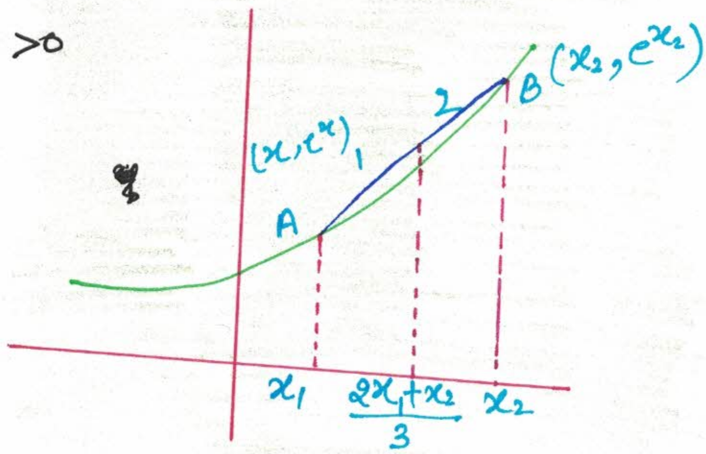
$$f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$$

H.P.

Question for any x_1, x_2 such that $x_1<x_2$ P.T. $\frac{2e^{x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$

$\rightarrow$ Let $f(x) = e^x$

$f'(x)>0,\ f''(x)>0$



$$y_{AB} = \frac{2e^{x_1}+e^{x_2}}{3}$$

$$y_c = e^{\frac{2x_1+x_2}{3}}$$

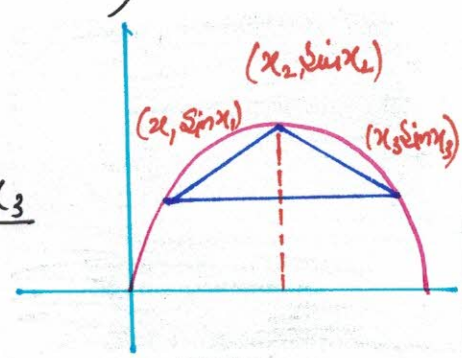
$$\boxed{y_{AB} > y_c}$$

Question If $0<x_1+x_2+x_3<\pi$, $x_1, x_2, x_3>0$

P.T. $\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin x_1+\sin x_2+\sin x_3}{3}$

$\rightarrow$ Let $f(x) = \sin x$

centroid of $\Delta = \frac{x_1+x_2+x_3}{3}, \left(\frac{\sin x_1+\sin x_2+\sin x_3}{3}\right)$



At $\frac{x_1+x_2+x_3}{3}$

$$\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin(x_1)+\sin(x_2)+\sin x_3}{3}$$

NATURE OF ROOTS OF CUBIC EQUATION

Let $f(x) = ax^3 + bx^2 + cx + d$

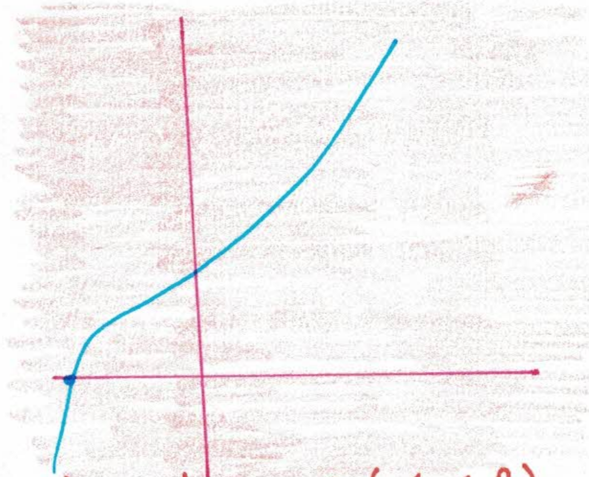
$f'(x) = 3ax^2 + 2bx + c$

CASE I

If $f'(x) > 0 \ \forall \ x \in R$

Then $f(x)$ has exactly one real root strictly ↑

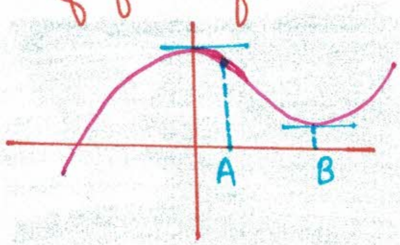
(Monotonic)



CASE II

If $f(x) = 0$ has two real roots α, β $(\alpha < \beta)$

if $f(\alpha) f(\beta) > 0 \Rightarrow f(x) = 0$ has one real root.



(Non monotonic)

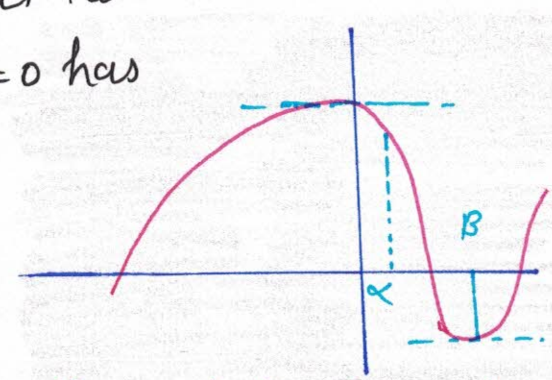
CASE III

If $f'(x) = 0$ has two distinct real roots α, β $(\alpha < \beta)$

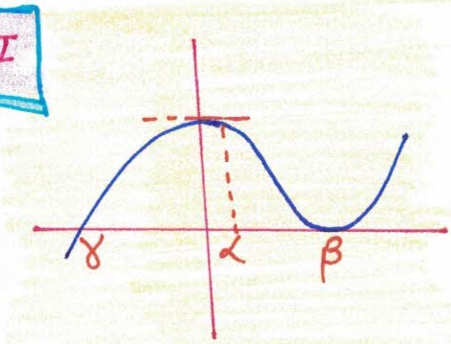
such that $f(\alpha) f(\beta) < 0$ then $f(x) = 0$ has

3 real roots

(Non Monotonic)



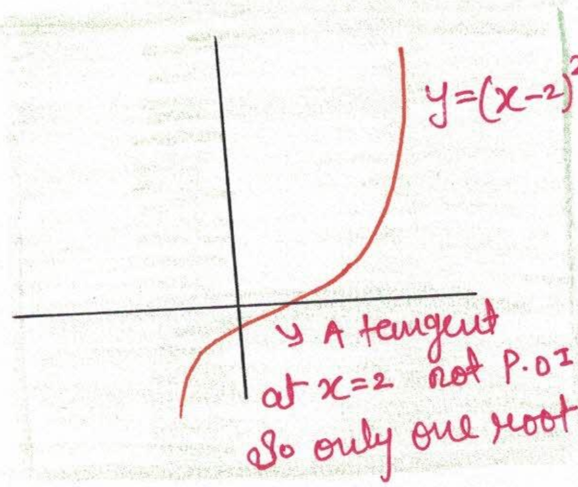
CASE IV

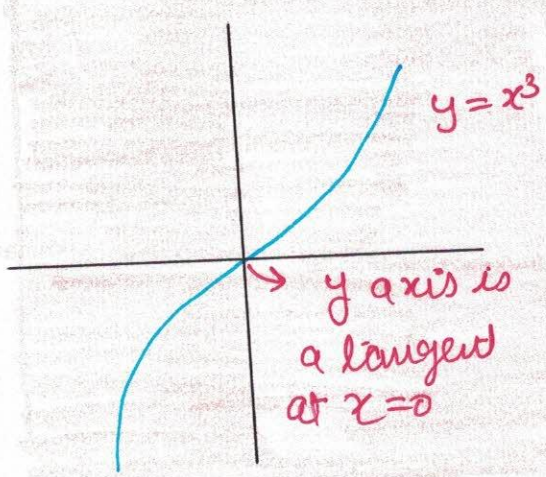


if $f'(x) = 0$ has two distinct real roots α, β $(\alpha < \beta)$ & $f(\beta) = 0$. then $f(x) = 0$ has one repeated root and one other root

$f(x) = (x - \beta)^2 (x - \gamma)$

CASE V If $f'(x)=0$ has equal roots, then $f(x)=0$ has Three equal roots and $f(\alpha)=0$





Question

Find the possible value of a for which $x^3+ax+4=0$ has one real root and $f(x)=x^3+ax+4$ is non monotonic.

$\rightarrow \quad x^3+ax+4=0$

$f'(x)=3x^2+a$

$f'(x)$ has two distinct real roots

$f'(x)=3x^2+a$

$\Delta>0, \quad f(\alpha)f(\beta)>0$

$0-4.3\,a>0$

$-12a>0$

$a<0$

$3x^2+a=0$

$x=\pm\sqrt{\frac{-a}{3}}$

$x=\pm\sqrt{\alpha}$

$f(-\sqrt{\alpha})f(\sqrt{\alpha})>0$

$\left(-(\sqrt{\alpha})^3+a\sqrt{\alpha}-4\right]\left[(\sqrt{\alpha})^3+a\sqrt{\alpha}+4\right]>0$

$-\left[\left((\sqrt{\alpha})^3+a(\sqrt{\alpha})^2-16\right]>0$

$$16-(\alpha^3+a^2\alpha+2a\alpha^2)>0$$

$$\alpha^3+a^2\alpha+2a\alpha^2-16<0$$

$$\frac{-a^3}{27}-\frac{a^3}{3}+\frac{2a^3}{9}-16<0$$

$$\frac{-a^3-9a^3+6a^3}{27}-16<0$$

$$\frac{4a^3}{27}+16>0$$

$$a>\left(\frac{-16+27}{4}\right)^{1/3} \qquad \boxed{a<0}$$

Question → If $f(x)=x^3+px+4=0$ has 3 real roots. Find P.

$$f'(x)=3x^2+P$$

$$3x^2+P=0 \Rightarrow x=\pm\sqrt{\frac{-P}{3}} \Rightarrow x=\pm\sqrt{\beta}$$

$$f(\alpha)\,f(\beta)<0$$

$$\beta^3+\beta^2\alpha+2P\beta^2-16>0$$

$$\frac{4P^3}{27}+16<0$$

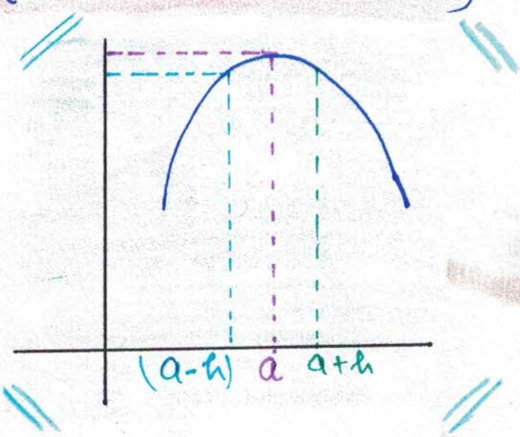
$$\boxed{P<\left(\frac{-16\times 27}{4}\right)^{\frac{1}{3}}}$$

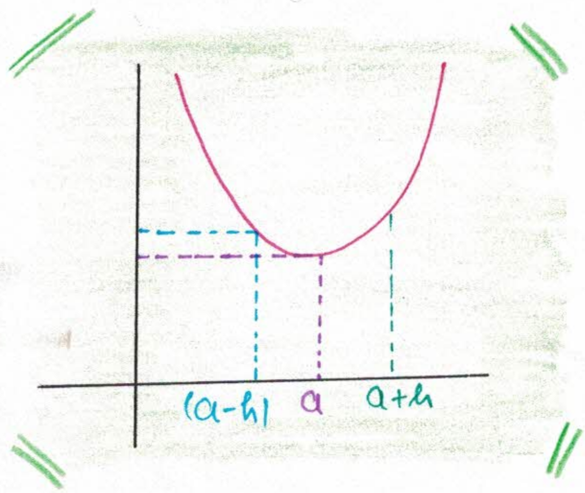
LOCAL MAXIMA / MINIMA

Let $f(x)$ be a function defined for an open interval containing at pt. 'a:

If $f(x)$ has a local maxima at $x=a$, then the value of $f(a)$ is greater than the value of $f(x)$ $\forall$ $x \in (a-h, a+h) - \{a\}$ for small +ve h.

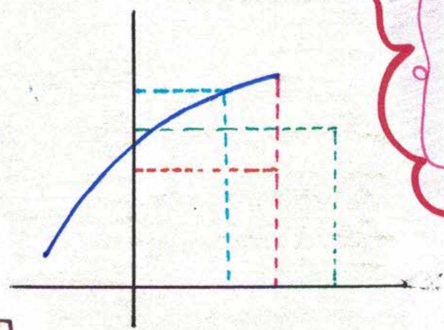
$$\left.\begin{matrix} f(a) > f(a-h) \\ f(a) > f(a+h) \end{matrix}\right\}$$ for any small +ve h then 'a' is local maxima.





If $f(x)$ has a local minima at $x=a$, then $f(a) < f(x)$ $\forall$ $x \in (a-h, a+h) - \{a\}$ i.e.

$$\left\{\begin{matrix} f(a-h) > f(a) \\ f(a+h) > f(a) \end{matrix}\right\}$$ then 'a' is called local minima



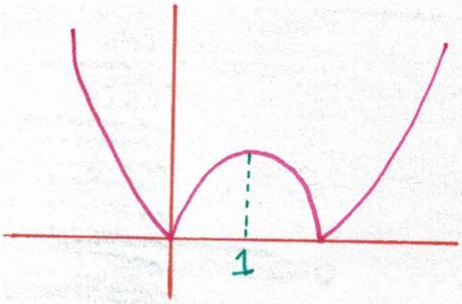
{continuity & differentiability are not necessary at $x = a$.}

Question

Find the pt-s of local maxima / minima for $f(x) = |x^2 - 2x|$

$f(x) = |x(x-2)|$

$$f(x) = \begin{cases} x(x-2), & x < 0 \\ -x(x-2), & x \in [0,2] \\ x(x-2), & x \in [2,\infty] \end{cases}$$



At $x = 0$, $\quad f(x) = 0$

$x = h$, $-h(h-2) = 2h - h^2 > (2-h)h > 0$

$x = -h$, $-h(-h,-2) = h(h+2) > 0$

$x = 0 \rightarrow$ local minima.

At $x = 2$, $f(x) = 0$

$x = 2+h$, $2+h(2+h-2) = (2+h)h > 0$

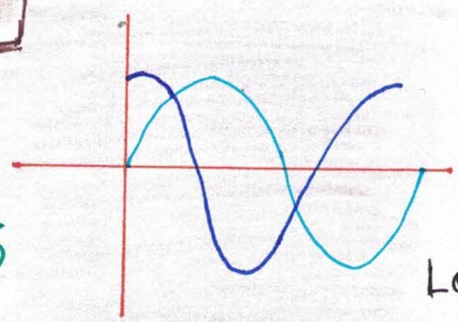
$x = 2-h$, $-(2-h)(2-h-2) = (2-h)h > 0$

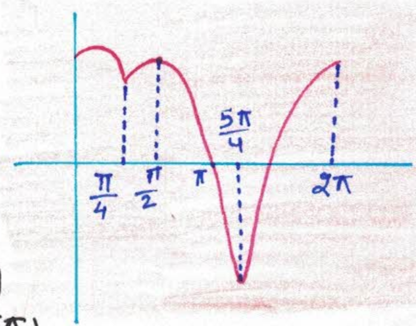
$x = 0 \rightarrow$ local minima

(maxima at $x = 1$)

Question

$f(x) = \max\{\sin x, \cos x\}$ in $(0, 2\pi)$





Local Max at $(\pi/2)$

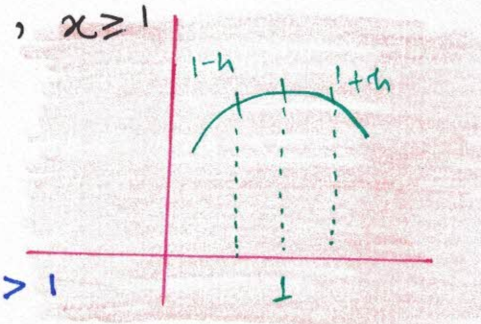
Local Min at $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Question

If $f(x) = \begin{cases} |x-1| + a^2 - 9a - 9, & x < 1 \\ 4x - 3, & x \geq 1 \end{cases}$

has local minima at $x = 1$ then find a.

$\rightarrow$ $f(x) = \begin{cases} 1 - x + a^2 - 9a - 9, & x < 1 \\ x - 1 + a^2 - 9a - 9, & x \geq 1 \\ 4x - 3, & x \geq 1 \end{cases}$



At $x = 1$, $4 - 3 = 1$

At $x = 1 + h$, $4(1+h) - 3 \geq 1$

At $x = 1 - h$, $1 - x + a^2 - 9a - 9 > 1$

$a^2 - 9a - 9 > 1$

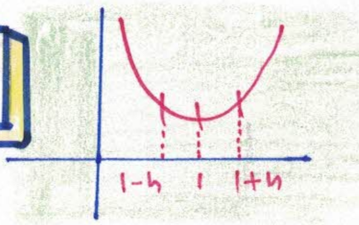
$a^2 - 9a - 10 > 0$

$(a - 10)(a + 1) > 0$

$f(1-h) < 1$

$f(1+h) < 1$

$\boxed{a \in (-\infty, -1) \cup (10, \infty)}$



Question

If $f(x) = \begin{cases} \cos\left(\frac{\pi}{2}x\right), & x \geq 0 \\ x + a, & x < 0 \end{cases}$

then find 'a' if f has local maxima at $x = 0$

$f(0) = \cos 0 = 1$

$f(-h) = -h + a \simeq a$

$\boxed{a < 1}$

Question

Find the pts. of local minima of

$f(x) = |x-1| + 2|x-2| + 3|x-3|$

$f(x) = -x + 1 - 2x + 4 - 3x + 9$

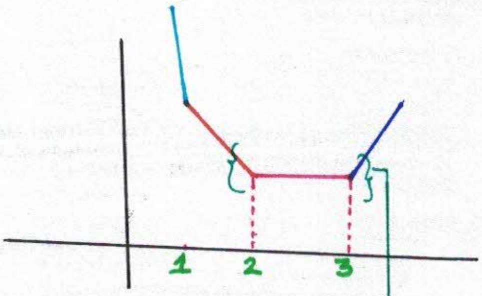
$= -6x + 14 \quad (x < 1)$

$$f(x) = x-1-2x+4-3x+9$$
$$= -4x+12 \quad (x \in (1,2))$$

$$f(x) = x-1+2x-4-3x+9$$
$$= 4 \quad x \in (2,3)$$

$$f(x) = x-1+2x-4+3x-9$$
$$= 6x-14 \quad (x>3)$$

not Needed



equal case also Possible

$\boxed{x = 2, 3 \text{ are local minima}}$

Question

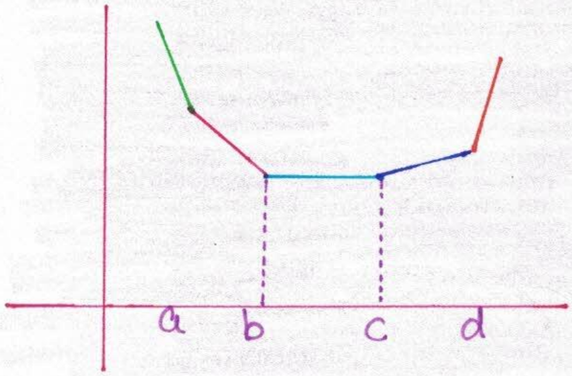
If $a<b<c<d$ then find pts. of extremum of
$f(x) = |x-a| + |x-b| + |x-c| + |x-d|$

$x=a$, $b+c+d-3a$

$x=b$, $b-a+c-b+d-b$
$= d+c-a-b$

$x=c$, $c-a+c-b+d-c$
$= d+c-a-b$

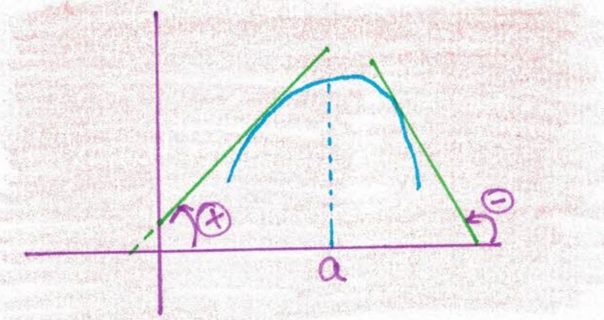
$x=d$, $3d-a-b-c$.



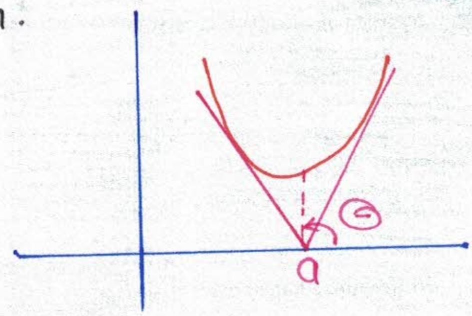
($x = b, c$ are pts. of minima)

DERIVATIVE TEST FOR LOCAL MAXIMA/MINIMA

Let $f(x)$ be a continuous function in an open interval containing a point 'a', then $f(x)$ has a local maxima at $x=a$ if $f(a-h) > 0$ & $f'(a+h) < 0$ for small +ve h.



A local minima at $x=a$ if $f'(a-h) < 0$ & $f'(a+h) > 0$ for small +ve h.



If Derivative of $f(x)$ changes its sign the neighbourhood of pt. 'a' from +ve to -ve then $f(x)$ has local maxima at $x=a$

PROOF

$$(a-h) \leq f(a)$$

$$f(a-h) - f(a) \leq 0$$

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} \geq 0$$

$$\boxed{f'(a) \geq 0}$$

$$f(a+h) \leq f(a)$$

$$f(a+h) - f(a) \leq 0$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \leq 0$$

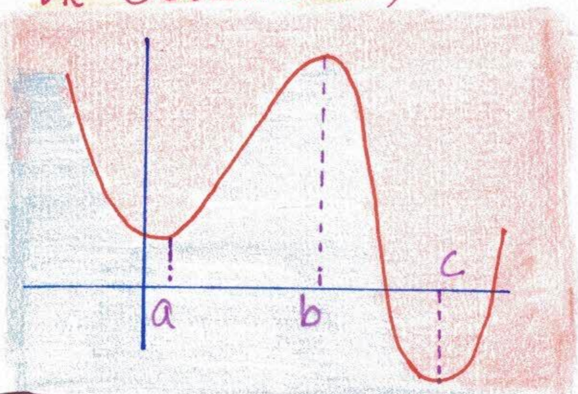
$$\boxed{f'(a) \leq 0}$$

If Derivative of $f(x)$ changes its sign from $-ve$ to $+ve$ in the neighbourhood of a point $x=a$, then $f(x)$ is the Pt. of local minima at $x=a$.

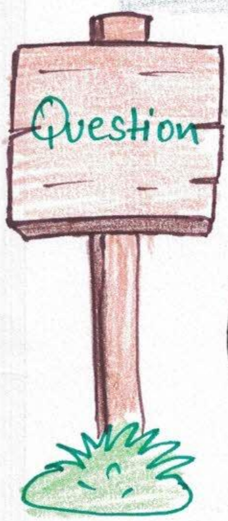
$$f'(a-h) < 0 \ \& \ f'(a+h) > 0$$

$f(x)$ decreases to increases in the neighbourhood of a.

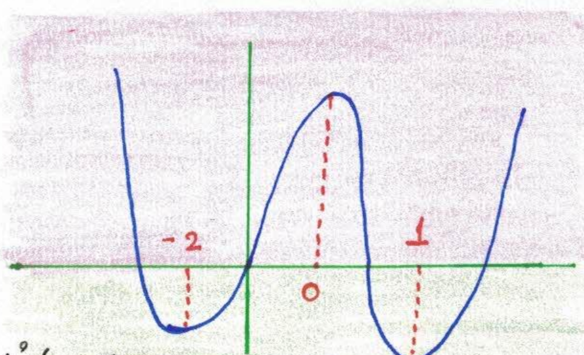
At extremum, the derivative of $f(x)=0$



There exist a Point of local maxima b/w two minima of $f(x)$.



If $f(x) = 3x^4 + 4x^3 - 12x^2 + a$ has 4 real roots then find a.



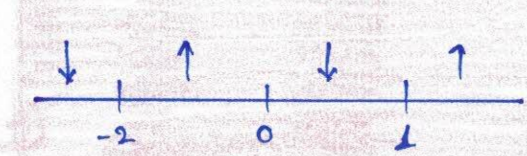
$\left\{ \begin{array}{l} \text{A maxima b/w two} \\ \text{minima } f'(x) = 12x^3 + 12x^2 - 24x \end{array} \right\}$

For minima $f'(x-h) < 0 \ \& \ f'(x+h) > 0$

At extremum Pt. $12x^3 + 12x^2 - 24x = 0$

$$12x(x^2 + x - 2) = 0 \Rightarrow 12x(x+2)(x-1)$$

$$x = (0, -2, +1)$$



$\Big\}$ Maxima > 0
minima < 0

Maxima at $x=0$, $f(x)=a>0$

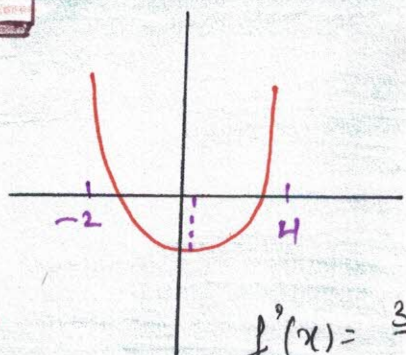
Minima at $x=1$, $f(x)=a-5<0 \Rightarrow a<5$

Minima at $x=-2$, $f(x)=a-32<0 \Rightarrow a<3$

$$\boxed{a \in (0,5)}$$

Question

If $f(x)=\frac{x^3}{3}-ax^2+(a^2-1)x-1$ has pts. of local minima & maxima b/w -2 & 4 then find a.



$f'(x)=0,\ x=(a-1 \text{ or } a+1)$

$\left.\begin{matrix} a-1=-2 \\ a+1=4 \end{matrix}\right] - (-1,3)$

$$f'(x)=\frac{3x^2}{3}-2ax+a^2-1 = x^2-2ax+a^2-1.$$

$$f'(x)=0 \Rightarrow x^2-2ax+a^2-1=0$$

↳ roots lie b/w $(-2, 4)$

$$f'(-2)>0 \Rightarrow 4+4a+a^2-1>0$$

$$a^2+4a+3>0$$

$$(a+1)(a+3)>0 \Rightarrow a \in (-\infty,-3)\cup(-1,\infty)$$

$$f'(4)>0 \Rightarrow 16-8a+a^2-1>0$$

$$a^2+4a+3>0$$

$$(a+5)(a+3)>0 \Rightarrow a \in (-\infty,+3)\cup(\pm 5,\infty)$$

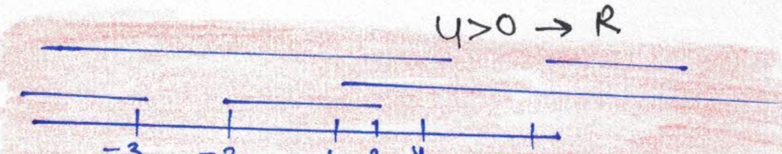
$$-2<\frac{-b}{2a}<4$$

$$-2<\frac{2a}{2\times 1}<4 \Rightarrow a \in (-2,4)$$

$D \geq 0$

$4a^2-4(a^2-1)>0$

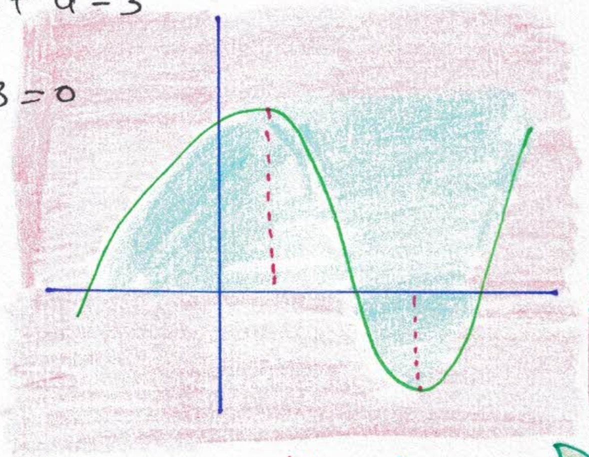
$4>0 \rightarrow R$



$$\boxed{a \in (-1,3)}$$

Question If $f(x) = x^3 - 3(a-7)x^2 + (a^2-3)x - 1$ has pt. of local maxima at some +ve value of x then find a.

$$f'(x) = 3x^2 - 6(a-7)x + a^2 - 3$$



$$3x^2 - 6(a-7)x + a^2 - 3 = 0$$

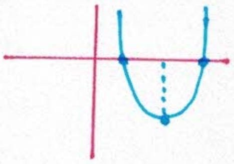
$$\Delta > 0$$

$f'(x)$ has both roots +ve

$$f'(0) > 0$$

$$a^2 - 3 > 0 \Rightarrow a^2 > 3$$

$$\Rightarrow a \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$$



$$\Delta > 0$$

$$12a^2 - 504a + 1800 > 0$$

$$a^2 - 42a + 150 > 0$$

$$a = \frac{42 \pm \sqrt{1764 - 600}}{2} = \boxed{\frac{42 + \sqrt{1164}}{2}}$$

Question Let $f(x)$ be a polynomial of degree 6 and has local maxima at $x = -1$ and 1 if $\lim_{h \to 0} \ln\left(1 + \frac{f(x)}{x^3}\right)^{1/x} = 2$ Then find $f(x)$.

Let $f(x) = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + \_\_\_\_.$

for limit to exist, 1^∞ must exist

$$\frac{f(x)}{x^3} \to 0$$

$f(x)$ will not have x^3, x^2, x & const. terms

$$f(x) = a_0x^6 + a_1x^5 + a_2x^4$$

$$f(x) = a_0x^6 + a_1x^5 + a_2x^4$$

$$\frac{a_0x^6 + a_1x^5 + a_2x^4}{x^3} \to 0$$

$$\ln e^{\left(\frac{f(x)}{x^3}\right)\times\frac{1}{x}} \Rightarrow 2$$

$$\ln e^{\frac{fx}{x^4}} = 2 \Rightarrow \frac{f(x)}{x^4} = 2$$

$$\boxed{a_2 = 2}$$

$f(x) = a_0x^6 + a_1x^5 + 2x^4$ $f'(-1) = 0$

$f'(x) = 6a_0x^5 + 5a_1x^4 + 8x^3$ $f'(1) = 0$

$f'(x) = 0$ at 1 & -1

$6a_0 + 5a + 8 = 0$

$6a_0 + 5a_1 = -8$

$-6a_0 + 5a_1 - 8 = 0$

$-6a_0 + 5a_1 = 8$

$a_1 = 0$ $a_0 = -\frac{4}{3}$

$$\boxed{f(x) = \frac{-4}{3}x^6 + 2x^4}$$

SECOND DERIVATIVE TEST

Let $y = f(x)$ be a continuous & diff^n function -

If $f''(x_0) > 0$, $f(x)$ has local minima at x_0

$f''(x_0) < 0$, $f(x)$ has local maxima at x_0

$f''(x_0) = 0$, This test fails.

N^{th} ORDER DERIVATIVE

If $f'''(x_0) = 0 = f''(x_0) = \ldots\ldots\ f^{n-1}(x_0) = 0$.

If $f^n(x_0) \neq 0$,

If n is even and $f^n(x_0) > 0$, then x_0 is pt. of minima.

If n is even and $f^n(x_0) < 0$, then x_0 is pt. of maxima.

If n is odd, then $x = x_0$ is neither pt. of max. or min.

eg: $f(x) = x^4$

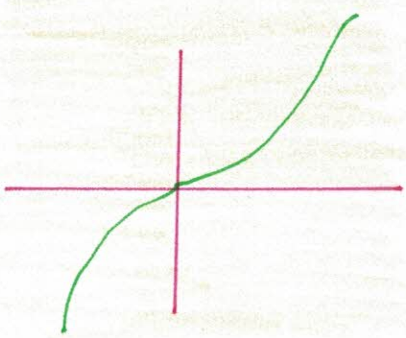
$\underbrace{f^{(4)}}_{\text{even}}(x) = 4! > 0$

$x = 0$ is pt. of min.

$f(x) = x^5$

$\underbrace{f^{(5)}}_{\text{odd}} = 5! > 0$

$x = 0$ is neither pt. of min or max.



$x = 0$ is saddle pt. (resting pt.)

Find local minima | maxima of

$$f(x) = \frac{50}{3x^4 + 8x^3 - 18x^2 + 60} \quad \forall\, x \in [-2, 5]$$

$\longrightarrow$ Let $g(x) = 3x^4 + 8x^3 - 18x^2 + 60$

$g'(x) = 12x^3 - 24x^2 - 36x$

$g'(x) = 0 \Rightarrow 12x(x^2 + 2x - 3) = 0$

$12x(x+3)(x-1$

$x = 0$,

$x = 0, 1, -3 \rightarrow$ Not in domain

$g''(x) = 36x^2 + 48x - 36$

At $x=0$, $g''(x) = -36 < 0 \rightarrow$ max.

At $x=1$, $g''(x) = 48 > 0 \rightarrow$ Min.

At $x=-3$, $g''(x) = 324 - 144 - 36 > 0 \rightarrow$ Min

$f(x)$ has local max at (1) and Min at $x=0$

OPERATION ON FUNCTION HAVING LOCAL MAX|MIN at $x=a$

If $f(x)$ has local min at $x=a$, then $-f(x)$ has local max. at $x=a$ and vice versa.

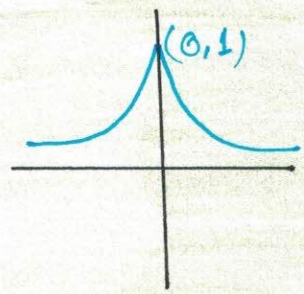
EX:- $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ at $x = \frac{3\pi}{4} = 1$ max

$= -\sin\left(x - \frac{\pi}{4}\right)$ at $x = \frac{3\pi}{4} = -1$ Min

If $f(x)$ & $g(x)$ has local maxima or minima at $x=a$ Then $f(x) + g(x)$ has local max or min at $x = a$

EX:- $h(x) = \cos x + e^{-|x|}$

max at $x=0$



$h(x)$ has local max. at $x=0$

COMPOSITION...

Let $f(x)$ and $g(x)$ be two continuous and diff^n function for.

Let $f(x)$ and $g(x)$ be two continuous and diff^n fcn.

If $f(x)$ is defined at $x=a$ and $g(x)$ has local max^m (min^m) at $f(a)$ then $g\,of(x)$ has also local max^m & min^m at $x=a$.

e.g. $f(x) = 1-x^3$ $\quad$ $g(x) = 2x-x^2$

$f(0) = 1 - \cancel{0^3}$ $\quad$ $g'(x) = 2-2x = 2(1-x)$

$g''(x)\big|_{x=1} < 0$

max. at $x=1$

$g(x)$ has max. at $f(0)=1$

$g\,of(x)$ has max. at $x=0$

$g\,of(x)$ local min^m of max^m are independent of the monotonicity of $f(x)$.

If $f(x)$ has local max^m / min^m at $x=a$ and $g(x)$ is an increasing function at $x=f(a)$ then $g\,of(x)$ has local max^m / min^m at $x=a$.

Ex:- $f(x) = 2x-x^2$ has local max. at $x=1$

$g(x) = \tan^{-1}(x)$

$g'(x) = \dfrac{1}{1+x^2} > 0 \longrightarrow \uparrow$ function

$g\,f(x) = \tan^{-1}(2x-x^2)$ also has local max. at $x=1$

Ex:- $f(x) = 2x-x^2 \longrightarrow$ max. at $x=1$

$g(x) = e^x \longrightarrow \uparrow$ function

$g(f(x))$ has local max. at $x=1$

If $f(x)$ has local max^m/min^m at $x = a$ and $g(x)$ is a decreasing function at $x = f(a)$ then $gof(x)$ has local min^m/max^m at $x = a$.

EX:- $f(x) = 2x - x^2$ has max at $x = 1$

$g(x) = \cot^{-1} x$

$g'(x) = \frac{-1}{1+x^2} < 0 \rightarrow \downarrow$ function

$gf(x) = \cot^{-1}(2x - x^2)$ has local min at $x = 1$

$$g'f(x)\Big|_{x=1} = \frac{-1 \times (2-2x)}{(2x-x^2)^2} = \frac{2(x-2)}{1+(2x-x^2)^2}$$

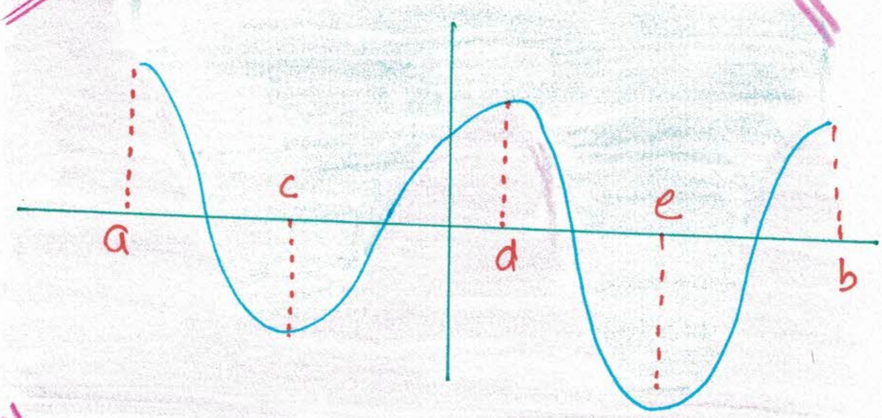
$$= \frac{2(x-2)}{1+x^2(2-x)^2} < 0$$

$\downarrow$

at $x = 1$

GLOBAL MAXIMA / MINIMA IN [a, b]

Let $f(x)$ be cont^n & diff^n for interval $[a, b]$



$f(x)$ has local maxima at $x = a, d, b$

$f(x)$ has local minima at $x = c, e$

$f(x) = 0 \Rightarrow x = c_1\, c_2 \ldots\ldots c_n \in [a, b]$

Let $M = \max\{f(c_1), f(c_2) \ldots\ldots f(c_n)\}$

$m = \min\{f(c_1), f(c_2) \ldots\ldots f(c_n)\}$

Then M is called global maxima.

m is called global minima.

GLOBAL MAXIMA / MINIMA IN (a, b)

$f(x) = 0 \Rightarrow x = c_1, c_2 \ldots\ldots c_n \in (a, b)$

Let $M = \max\{f(c_1), f(c_2) \ldots\ldots f(c_n)\}$

$m = \min\{f(c_1), f(c_2) \ldots\ldots f(c_n)\}$

If $M > \lim\limits_{x\to a^+} f(x)$ & $\lim\limits_{x \to b^-} f(x) \longrightarrow$ global max.

If $M < \underbrace{\lim\limits_{x\to x^+} f(x)}_{\hookrightarrow\, l_1}$ + $\underbrace{\lim\limits_{x\to b^-} f(x)}_{\hookrightarrow\, l_2}$ then global max^m does not exist

If $m < l_1$ + l_2, then $m \longrightarrow$ global minima

If $m > l_1$ + l_2, then global minima does not exist.

Question

Find the greatest value of $\frac{x^3}{3} - \frac{5}{2}x^2 + 6x - 7$ in $[1, 5]$

$$f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 6x - 7$$

$$f'(x) = x^2 - 5x + 6 = 0$$
$$= (x-3)(x-2) = 0$$
$$x = 2, 3$$

$$f''(x) = 2x - 5$$
$$f''(x)\Big|_{x=2} = -1 < 0$$
$$f''(x)\Big|_{x=3} = 1 > 0$$

At $x=2 \Rightarrow \frac{8}{3} - \frac{5}{2} \times 4 + 6 \times 2 = 7$

$$= \frac{8}{3} - 10 + 12 - 7 = \frac{8}{3} - 5 = \frac{-7}{3}$$

At $x=5$, max $= \frac{125}{3} - \frac{5}{2} \times 25 + 30 - 7 = 42.3 - 62.5 + 23$

$$= 65.3 - 62.5$$
$$= 2.8$$

Max. at $x=5$

Question

Find the global max^m/ min^m of $f(x) = x^3 - 4x^2 - 7x + 6$ in $[-3, 0]$ in $(0, 2)$.

$$f'(x) = 3x^2 - 8x - 7 = 0$$
$$x = \frac{8 \pm \sqrt{64 + 12 \times 7}}{6} = \frac{8 \pm \sqrt{148}}{6}$$
$$= \frac{8 \pm 12.2}{6} = \frac{20.2}{6}, \frac{-4.6}{6}$$
$$= 3.36, -0.7$$

3.36 → Out of domain

$$f(0) = 6$$
$$f(-0.7) = (-0.7)^3 - 4(0.7)^2 - 7(0.7) + 6$$
$$= -0.343 - 0.196 - 0.49 + 6 < 6$$

Global max. at $x=0$ as $f(0) > f(0^+)$ & $f(2^-)$

Question

$$f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ 3 - 2x + \log_2(a^2-4), & x > 1 \end{cases}$$ has greatest value at $x = 1$ then find a.

$$f(1) = 1 - 1 + 10 - 5 = 5$$

$$f(1^+) = 3(-2)(1+h) + \log_2(a^2-4)$$
$$= 3 - 2 - 2h + \log_2(a^2-4)$$
$$= \log_2(a^2-4) + 1$$

$$\log_2(a^2-4) + 1 \le 5$$
$$\log_2(a^2-4) \le 4$$
$$a^2 - 4 \le 16$$
$$a^2 \le 20$$

$$a^2 - 4 > 0$$
$$a^2 > 4$$
$$a \in (-\infty, -2) \cup (2, \infty)$$

$$a^2 - 20 \le 0$$
$$(a + 2\sqrt{5})(a - 2\sqrt{5}) \le 0$$

$$a \in [-2\sqrt{5}, 2\sqrt{5}] \cap (-\infty, -2) \cup (2, \infty)$$

$$a \in [(-2\sqrt{5}, -2) \cup (2, 2\sqrt{5})]$$

MAX^m / MINI^m OF FUNCTION OF TWO VARIABLE UNDER ANY GEOMETRICAL RESTRICTION

$f(x,y)\Big|_{max/min}$ under geometrical restriction that $f(x,y)$ lies on a curve.

We assume x, y as parametric forms and find the max / min value by putting x, y in that curve.

Question

If $f(x,y) = x^2 + 3y^2 - 3xy + 7$. Find $F_{max}/_{min}$ under the restriction $x^2 + 4y^2 = 4$.

$\rightarrow$

$$x^2 + 4y^2 = 4$$

$$\frac{x^2}{4} + \frac{4y^2}{4} = 1$$

$$\frac{x^2}{2^2} + \frac{y^2}{(1)^2} = 1$$

Paramatic form : $(2\cos\theta, \sin\theta)$

$$f(x,y) = 4\cos^2\theta + 3\sin^2\theta - 6\cos\theta\sin\theta + 7$$

$$= 3 + \cos^2\theta - 3\sin 2\theta + 7$$

$$= 3 + \frac{1 + \cos 2\theta}{2} - 3\sin 2\theta + 7$$

$$= 3 + \frac{1}{2} + \frac{\cos 2\theta}{2} - 3\sin 2\theta + 7$$

$$= \frac{21}{2} + \left[-\sqrt{\frac{1}{4}+9}, \sqrt{\frac{1}{4}+9}\right]$$

$$\boxed{f(x,y) = \left[\frac{21}{2} - \frac{\sqrt{37}}{2}, \frac{21}{2} + \frac{\sqrt{37}}{2}\right]}$$

Question

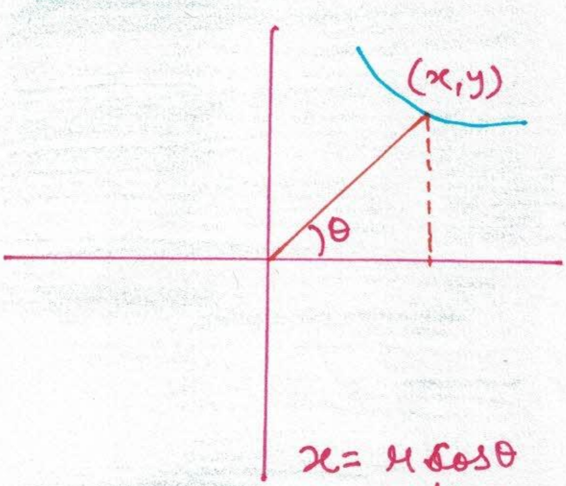
If $f(x,y) = x^2 + y^2$ then find $f_{max}|_{min}$ under the restricting $ax^2 + by^2 + 2hxy = 1$.

$\rightarrow$ $ax^2 + by^2 + 2hxy = 1$

(x,y) is any Point in Space

$$r^2(a\cos^2\theta + b\sin^2\theta + h\sin 2\theta) = 1$$

$$r^2 = \frac{1}{a\cos^2\theta + b\sin^2\theta + h\sin 2\theta}$$



$x = r\cos\theta$

$y = r\sin\theta$

$a\cos^2\theta + b\sin^2\theta + h\sin 2\theta$

$= a\left(\frac{1+\cos 2\theta}{2}\right) + b\left(\frac{1-\cos 2\theta}{2}\right) + h\sin 2\theta$

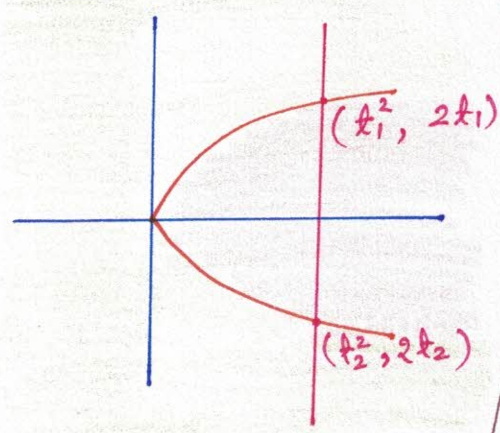
$= \left(\frac{a}{2} + \frac{b}{2}\right) + \left(\frac{a}{2} - \frac{b}{2}\right)\cos 2\theta + h\sin 2\theta$

$= \left(\frac{a+b}{2}\right) + \left[-\sqrt{\left(\frac{a-b}{2}\right)^2 + h^2}, \sqrt{\left(\frac{a-b}{2}\right)^2 + h^2}\right]$

$f(x,y) \longrightarrow \max = \left[\frac{a+b}{2} + \sqrt{\left(\frac{a-b}{2}\right)^2 + h^2}\right]^{-1}$

$\longrightarrow \min = \left[\left(\frac{a-b}{2}\right) - \sqrt{\left(\frac{a-b}{2}\right)^2 + h^2}\right]^{-1}$

Question Find the shortest normal chord to the curve $y^2 = 4x$.



$\longrightarrow t_2 = t_1 - \frac{2}{t_1}$

$\text{Dist}^n = \sqrt{(t_1^2 - t_2^2)^2 + 4(t_1 - t_2)^2}$

$= \sqrt{(t_1 - t_2)^2\{(t_1 + t_2)^2 + 2\}}$

$(\text{Dist})^2 = (t_1 - t_2)^2\{(t_1 + t_2)^2 + 4\}$

$= \left(t_1 + t_1 + \frac{2}{t_1}\right)^2\left\{\left(t_1 - t_1 - \frac{2}{4}\right)^2 + 4\right\}$

$= \left(2t_1 + \frac{2}{t_1}\right)^2\left\{\left(\frac{4}{t_1^2}\right) + 4\right\}$

$= \frac{(2t_1^2 + 2)^2\ (4 + 4t_1^2)}{t_1^2}$

$$= 4\left\{\frac{(t_1^2+1)\,(1+t_1^2)}{t_1^2}\right\} = 4\left\{\frac{(t_1^4+1+2t_1^2)\,(1+t_1^2)}{t_1^2}\right\}$$

$$f(x) = \text{Dist.} = \sqrt{4\left\{\left(t_1^2+\frac{1}{t_1^2}+2\right)\left(\frac{1}{t_1^2}+1\right)\right\}}$$

$$f(x) = \frac{\frac{1}{2\sqrt{4\left(t_1^2+\frac{1}{t_1^2}+2\right)\left(\frac{1}{t_1^2}+1\right)}} \times 4\left(\frac{1}{t^2}+1\right)\left(2t_1-\frac{2}{t_1^3}\right)+\left(t_1^2+\frac{1}{t_1^2}+2\right)\left(\frac{-2}{t_1^3}\right)}{} = 0$$

$$= 4\left(\frac{1}{t^2}+1\right)\left(2t_1-\frac{2}{t_1^3}\right)+\left(t_1^2+\frac{1}{t_1^2}+2\right)\left(\frac{-2}{t_1^3}\right) = 0$$

$$= 4\left\{\frac{2}{t_1}-\frac{2}{t_1^5}+2t_1-\frac{2}{t_1^3}-\frac{2}{t_1}-\frac{2}{t_1^5}-\frac{4}{t_1^3}\right\}$$

$$= \frac{-2+2t_1^6-2t_1^2-2-4t_1^2}{t_1^5} = 0$$

$$\Rightarrow -1+t_1^6-t_1^2-1-4t_1^2=0$$

$$t_1^6-3t_1^2-2=0$$

$$\boxed{t_1 = \sqrt{2}}$$

$$t_2 = -2-\frac{2}{\sqrt{2}} = -2-\sqrt{2} \qquad t_2^2 = 4+2+4\sqrt{2}$$

$$\text{Dist} = \sqrt{(t_1^2-t_2^2)^2+4(t_1-t_2)^2} = \sqrt{(2-6-4\sqrt{2})^2+4(2\sqrt{2}+2)^2}$$

$$= \sqrt{16+32+32\sqrt{2}+37+16+32\sqrt{2}}$$

$$\boxed{= \sqrt{96+64\sqrt{2}}}$$

If $f(x)$ is continuous in $[a, b]$ and $f(x)$ has range $[m, M]$ if $g(x)$ is continuous in $[m, M]$ then–

$$gf(x)\Big|_{\substack{max \\ in\, x \in [a,b]}} = g(x)\Big|_{\substack{max \\ in\, x \in [m,M]}}$$

$$gf(x\Big|_{\substack{min \\ in\, x \in [a,b]}} = g(x)\Big|_{\substack{min \\ in\, x \in [m,M]}}$$

Question

Find max^m and min^m Value :-

If $f(x) = \dfrac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$ in $\left[0, \frac{\pi}{2}\right]$

$\rightarrow$ $f(x) = \dfrac{\sin 2x}{\sin\left(2 + \frac{\pi}{4}\right)} = \dfrac{1 + 2\sin x \cos x - 1}{\sin\left(x + \frac{\pi}{4}\right)}$

$$= \frac{(\sin x + \cos x)^2 - 1}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{2\left(\sin\left(x + \frac{\pi}{4}\right)\right)^2 - 1}{\sin\left(x + \frac{\pi}{4}\right)}$$

$$\Rightarrow \frac{\sqrt{2}\,(\sin x + \cos x)^2 - 1}{\sin x + \cos x}$$

$g(x) = \sqrt{2}\dfrac{(x^2 - 1)}{x}$

$h(x) = \sin x + \cos x$

$= \sqrt{2}\left(\sin\left(x + \frac{\pi}{4}\right)\right)$

$= \sqrt{2}\left[\sin\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)\right]$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}}, 1\right] = \left[1, \sqrt{2}\right]$

$h(x)$- has max at $\sqrt{2}$

$h(x)$ has min at 1.

$g(x)_{max} = \sqrt{2}\left(\frac{2-1}{\sqrt{2}}\right) = 1$

$g(x)_{min} = \sqrt{2}\ \frac{(1-1)}{1} = 0$

$gh(x)$ has max. 1 & min 0

$$\boxed{gh(x) \in [0,1]}$$

Question

Find Max. and min of $f(x)$ $f(x) = \sin(\cos(\sin x))$ in $\left[\frac{\pi}{2}, \pi\right]$.

$f(x) = \sin(\cos(\sin x))$

$g(x) = \sin x$

$h(x) = \cos(\sin x)$

$h'(x) = -\sin(\sin x)\cos x = 0$

$x = \frac{\pi}{2}, \pi$

$h(x)\Big|_{\frac{\pi}{2}} = \cos 1$ $\qquad h(x)\Big|_{\pi} = 1$

$h(x)$ max $= 1$

$h(x)$ Min $= \cos 1$

$g(x)\Big|_{1} = \sin 1 \downarrow$ Max.

$g(x)_{\cos} = \sin(\cos 1) \rightarrow$ Min

$$\boxed{\begin{aligned} gh(x)\ \text{max} &= f(x) = \sin 1 \\ gh(x)\ \text{min} &= f(x) = \sin(\cos 1) \end{aligned}}$$

MAXIMUM / MINIMUM USING DISCRETE FUNCTION

Ex:- Find the largest term in the sequence :

1. $a_n = \dfrac{n}{n^2+10}$, $n \in N$

2. $a_n = \dfrac{n^2}{n^3+200}$

→ we assume all $a_n = \dfrac{x}{x^2+10} = f(x)$

$$f'(x) = \frac{(x^2+10) - x(2x)}{(x^2+10)^2} = 0$$

$$x^2+10-2x^2=0$$

$$x^2 = 10 \Rightarrow x = \pm\sqrt{10} = \pm 3.2$$

At $[3.2] = 3$, $a_n = \dfrac{3}{9+10} = \dfrac{3}{19}$

$[3.2]+1 = 4$, $a_n = \dfrac{4}{16+10} = \dfrac{4}{26} = \dfrac{2}{13}$

$$\boxed{\text{Max } a_n = \frac{3}{19}}$$

→ we assume $a_n = f(x) = \dfrac{x^2}{x^3+200}$

$$f'(x) = \frac{(x^3+200)(2x) - x^2(3x^2)}{(x^3+200)^2} = 0$$

$$2x^4 + 400x - 3x^4 = 0$$

$$x^4 = 400x \Rightarrow x^4 - 400x = 0$$

$$x(x^3-400)=0$$

$$x \neq 0,\quad x = \sqrt[3]{400} = 7.3$$

$$\text{At } [7.3] = 7,\quad a_n = \frac{49}{343+200} = \frac{49}{543}$$

$$= 8,\quad a_n = \frac{64}{512+200} + \frac{64}{712} = \frac{32}{306} = \frac{16}{153}$$

$$\boxed{\text{Max } a_n = \frac{49}{543}}$$

Question

In how many parts a +ve integral ≥ 5 be dissected such that product of all parts becomes max.

$\rightarrow$ $N = x_1 + x_2 + x_3 + \cdots\cdots x_n$

$$\text{Product} = (x_1 x_2 x_3 \cdots\cdots x_n) \longrightarrow \max$$

$$\frac{x_1 + x_2 + x_3 + \cdots\cdots x_n}{n} \geq (x_1 x_2 \cdots\cdots x_n)^{1/n}$$

$$\frac{N}{n} \geq (x_1 x_2 \cdots\cdots x_n)^{\frac{1}{n}}$$

$$\left(\frac{N}{n}\right)^n \geq x_1 x_2 \cdots\cdots x_n$$

$$f(x) = \left(\frac{N}{x}\right)^x \Rightarrow \log f(x) = x \log \frac{N}{x}$$

$$\frac{1}{f(x)} f'(x) = x \times \frac{x}{N} \times \frac{N}{x^2} + \log \frac{N}{x} = \log \frac{N}{x} - 1$$

$$f'(x) = \left(\frac{N}{x}\right)^2 \left[\log \frac{N}{x} - 1\right] = 0$$

$$\log \frac{N}{x} - 1 = 0 \Rightarrow \log \frac{N}{x} = 1$$

$$\frac{N}{x} = e \Rightarrow x = \frac{N}{e}$$

$$x = \left[\frac{N}{e}\right], \left[\frac{N}{e}\right] + 1$$

Question In how many parts 2014 be dissected such that Product of all parts be max.

$$\rightarrow \left[\frac{2014}{e}\right] \text{ or } \left[\frac{2014}{e}\right] + 1$$

$$= 741 \text{ or } 742$$

$$f(x) = \left(\frac{N}{n}\right)^n = \left(\frac{2014}{741}\right)^{741}$$

$$= \left(\frac{2014}{742}\right)^{742}$$

Let $g(x) = \left(\frac{2014}{x}\right)^x$

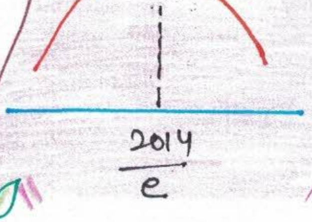
$$\log g(x) = x \log\left(\frac{2014}{x}\right)$$

$$\frac{1}{g(x)} g'(x) = -x \times \frac{x}{2014} \times 2014 \times \frac{1}{x^2} + \log\frac{2014}{x}$$

$$g'(x) = \left(\log\frac{2014}{x} - 1\right)\left(\frac{2014}{x}\right)^x$$

$$\log\frac{2014}{x} - 1 > 0$$

$$\log\frac{2014}{x} > 1 \Rightarrow \frac{2014}{x} > e \Rightarrow x < \frac{2014}{e}$$



$$\boxed{\text{Max. Parts} = 741}$$

MAX^m / MINI^m USING AM / GM / HM INEQUALITY

EX:- Find the max^m value of $\frac{x}{ax^2+b}$, $a, b, x > 0$

$$\rightarrow f(x) = \frac{x}{ax^2+b} = \frac{1}{ax + \frac{b}{x}}$$

$$\frac{xa + \frac{b}{x}}{2} \geq \left(ax \times \frac{b}{x}\right)^{\frac{1}{2}}$$

$$ax + \frac{b}{x} \geq 2\sqrt{ab}$$

$$\boxed{f(x)\Big|_{max} = \frac{1}{2\sqrt{ab}}}$$

Question

Find the min of $f(x) = \frac{x^2+x+2}{x}$

$$f(x) = \frac{x^2+x+2}{x} = x+1+\frac{2}{x} = x+\frac{2}{x}+1$$

$$\frac{x+\frac{2}{x}}{2} \geq \left(x \times \frac{2}{x}\right)^{1/2} \Rightarrow x+\frac{2}{x} \geq 2\sqrt{2}$$

$$\boxed{f(x)_{min} = 2\sqrt{2}+1}$$

Question

Find the min^m value of $f(x) = \frac{x^3+x+2}{x}$

$$f(x) = \frac{x^3+x+2}{x} \Rightarrow x^2+\frac{x}{x}+\frac{2}{x} = x^2+\frac{2}{x}+1$$

$$\Rightarrow x^2+\frac{1}{x}+\frac{1}{x}+1 \Rightarrow \frac{x^2+\frac{1}{x}+\frac{1}{x}}{3} > (1)^{1/3}$$

$$\Rightarrow x^2+\frac{2}{x} \geq 3 \quad \boxed{f(x)\Big|_{min} = 4}$$

Question Find Max^m value of x^2y^3 if $x+y=4$

$$\rightarrow \frac{\frac{x}{2}+\frac{x}{2}+\frac{y}{3}+\frac{y}{3}+\frac{y}{3}}{5} \geqslant \left(\frac{x^2y^3}{4\times 27}\right)^{\frac{1}{5}}$$

$$\frac{4}{5} \geqslant \frac{(x^2y^3)^{1/5}}{(4\times 27)^{1/5}} \Rightarrow \boxed{x^2y^3 \leq \left(\frac{4}{5}\right)^5 \times 4\times 27}$$

Question Find the min^m value of :—

$$f(x) = 2\tan^2 x + 18\cot^2 x + 7 \cdot \left(0, \frac{\pi}{2}\right)$$

$$f(x) = 2\tan^2 x + \frac{18}{\tan^2 x} + 7$$

$$= 2\left[\tan^2 x + \frac{9}{\tan^2 x}\right] + 7$$

$$\tan^2 + \frac{9}{\tan^2 x} \geqslant 2(9)^{1/2} \geqslant 6$$

$$= 2\times 6 + 7 = \boxed{19}$$

Question $f(x) = 2\cos^2 x + 8\sec^2 x - 5 \quad \left(0, \frac{\pi}{2}\right)$

$$f(x) = 2\cot^2 + 8\sec^2 x - 5$$

↳ cannot directly apply AM, GM

because if equality exists then $\cos x = \sqrt{2}$ not possible.

$$= 2\cos^2 x + 2\sec^2 x + 6\sec^2 x - 5$$

$$= 2\left[\cos^2 x + \frac{1}{\cos^2 x}\right] + 6\sec^2 x - 5$$

$$= 2\times 2 + 6 - 5 \Rightarrow 4 + 6 - 5 \Rightarrow \boxed{5}$$

Question If $a+b+c=4$ then find $(a, b, c \geq 0)$ max/min value of $ab+bc+ca$.

$$\frac{a+b+c}{3} \geq (abc)^{1/3}$$

$$\left(\frac{4}{3}\right)^3 \geq abc$$

$$abc|_{max} = \left(\frac{4}{3}\right)^3 \Rightarrow \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \left(\frac{1}{abc}\right)^{\frac{1}{3}}$$

$$\frac{ab+bc+ca}{3abc} \geq \left(\frac{1}{abc}\right)^{1/3}$$

$$ab+bc+ca \geq 3(abc)^{2/3}$$

$$ab+bc+ca \geq 3 \times \left(\frac{4}{3}\right)^{\cancel{3} \times \frac{2}{\cancel{3}}} = 3 \times \frac{16}{9} = \frac{16}{3}$$

$$\boxed{ab+bc+ca \geq \frac{16}{3}}$$

Question Find max/min of $f(x) = x^{3/2} + x^{-3/2} - 4\left(x+\frac{1}{x}\right)$ for all possible value of x.

$$f(x) = \left(x^{3/2} + \frac{1}{x^{3/2}}\right) - 4\left(x+\frac{1}{x}\right)$$

$$= \left(x^{1/2}\right)^3 + \left(x^{-1/2}\right)^3 - 4\left[\left(x^{1/2}+x^{-1/2}\right)^2 - 2\right]$$

$$x^{1/2} + x^{-1/2} = t.$$

$$f(x) = \left(x^{1/2}+x^{-1/2}\right)^3 - 3\left(x^{1/2}+x^{-1/2}\right) - 4\left(x^{1/2}+x^{-1/2}\right) + 8$$

$= t^3 - 3t - 4t^2 + 8$

$f'(x) = 3t^2 - 3 - 8t = 0$

$\Rightarrow 3t^2 - 9t + t - 3 = 0$

$3t(t-3) + (t-3) = 0 \quad \Rightarrow t = 3 \;, -\frac{1}{3}$

$f''(x) = 6t - 8$

At $t = 3 > 0 \longrightarrow$ min

At $t = -\frac{1}{3} < 0 \longrightarrow$ max

$f(x)\Big|_{min} = 27 - 3\times3 - 4\times9 + 8$

$= 27 - 9 - 36 + 8$

$= 35 - 45$

$$\boxed{f(x)\Big|_{min} = -10}$$